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Reflective Conversation and Knowledge Development in Pre-service Teachers: The Case of Mathematical Generalization

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Abstract

This article examines the development of professional knowledge in pre-service mathematics teachers. From the discussion of a task associated with the multiplication of consecutive integer numbers, generalization is recognized as a process that allows to explore, to explain, and to validate mathematical results, and as an essential ability to develop in the teaching of arithmetic, algebra, and geometry by using various representations. The study was conducted during three sessions of a didactic mathematics course, in which the instructor and ten students participated. The conversations of each session were recorded on audio and video. It was found that the reflective conversation fosters this knowledge from questions about the nature and processes of construction of mathematical concepts from a mathematical and didactic point of view.

Introduction

In the field of Mathematics Education, the teacher's learning and professional knowledge have been established as central aspects for both the improvement of the teaching practice and the professional development of the teacher. In recent decades, several studies have converged on the conclusion about the essential role of collectivity and reflection in teacher learning (Arcavi, 2016; Chamoso, Cáceres & Azcárate, 2012; Jaworski, 2006; Krainer & Llinares, 2010; Ponte, 2012; Preciado-Babb et al., 2015; Rasmussen, 2016; Rowland & Ruthven 2011; Santagata & Guarino, 2011; Thanheiser, Browning, Edson, Lo, Whitacre, Olanoff, Morton, 2014; Zaslavsky, 2008). In particular, it is recognised that learning to teach requires the teachers to develop different types of knowledge, because they must deeply understand the ideas that are intended to be taught and understand them in different ways (Shulman, 2005).

There are currently several theoretical proposals on the type of professional knowledge that teachers should have in order to teach mathematics efficiently. These proposals are based on Shulman (1986) who suggests that teachers require knowledge that goes beyond a general pedagogy to teach mathematics; that is, a pedagogy related to the specificity of the content to be taught. In this sense, the problematisation of the Mathematical Knowledge of the Teacher has been at the centre of proposals in mathematical education; some of these proposals are “Mathematical Knowledge for Teaching” (e.g. Ball, Thames, & Phelps, 2008), “proficiency in teaching mathematics" (e.g. Shoenfeld & Kilpatrick, 2008; Shoenfeld, 2011), "Didactic Mathematical Knowledge" (e.g. Pino-Fan, Godino, & Font, 2018), "Cognitive Activation in the classroom project - COACTIV- (e.g. Krauss, Neubrand, Blum, & Baumert, 2008; Kunter et al., 2013), "Didactic Knowledge of Mathematics teachers" (e.g. Ponte & Chapman, 2008; Llinares, 2012) and "Specialized Mathematical Knowledge of the Teacher" (e.g. Carrillo-Yañez et al., 2018).

Even when progress has been made on the subject of professional knowledge of mathematics teachers, "there is limited understanding of what it is, how one might recognize it, and how it might develop in the minds of teachers" (Silverman & Thompson, 2008, p. 2). Moreover, teaching mathematics demands several aspects related to their practice that cannot be solved by giving them knowledge (Shoenfeld, 2011). In fact, a central matter in this topic is "whether mathematical knowledge in teaching is located 'in the head' of the individual teacher or is somehow a social asset, meaningful only in the context of its application" (Rowland & Ruthven, 2011, p. 3). In the opinion of Rowland and Ruthven (2011), we are still far from reaching a consensus regarding a theoretical positioning that describes knowledge for teaching mathematics.

Most of the proposals on the knowledge of the teacher for teaching mathematics highlight the type of knowledge required by the teacher; however, little has been investigated on how they acquire this type of knowledge. In this sense, Thanheiser et al. (2014) argue that little is known about how future teachers learn. Hence our interest in providing information to answer this question by looking at how the professional knowledge of future mathematics teachers develops in contexts of social interaction such as the classroom, because we think that this is a social learning space that allows future teachers to learn how to teach collectively, and in our opinion, conversation and reflection are instruments that enhance these opportunities.

It is known that conversation is a process of negotiation of understandings, or agreements, in which participants may or may not agree, but they will always recognize a new thought (Pask, 1976; Scott, 2001), and reflection helps to integrate knowledge, understanding and experiences of learning of future teachers (Kaminski, 2003), which is reflected in the development of more competence for teaching mathematics. The study of Toom, Husu and Patrikainen (2015) shows that reflections among future teachers on the teaching of disciplinary content and the practice produce new understanding and contribute to the construction of knowledge of the practice. In this order of ideas, we analyze how the reflective conversation, among future teachers, contributes to the development of their professional knowledge, particularly about the concept of mathematical generalization. This interest is based on the fact that the collective and the reflective are key to the process of learning of the teacher. Horn and Little (2010) mention that collective learning is supported by the construction of shared meanings, trust, and conversation around work.

The professional learning community models – professionals coming together in a group (a community) to learn (Hord, 2008, p. 10) and inquiry communities (Jaworski, 2006) – produce, through conversation, new ideas that can lead to reflection; these are key aspects for teachers to potentially have opportunities to learn and develop skills aimed towards improving their teaching practice (Darling-Hammond et al., 2009). However, it is recognised that it is still necessary to specify the operability of these models and the measurement of their results (Campbell & Stohl, 2017). Also, Smith (2015) highlights that collective reflection is not as common as it should be expected in these communities. It is therefore necessary to expand research on how to promote meaningful reflection among teachers (Saylor & Johnson, 2014).

Generalization in Teaching Mathematics

Generalization is one kind of knowledge for teaching mathematics (Demonty, Vlassis, & Fagnant, 2018). Its importance lies mainly in two senses: as a teaching route to develop mathematical concepts (Davydov, 1990; Dörfler, 1991) and as an activity to support the development of algebraic thinking and the study of patterns in algebra (Radford, 2014; Warren, Trigueros, & Ursini, 2016; Zazkis & Liljedahl; 2002) and geometry (Moss & London, 2011). In the first sense, Davydov argues that the emergence of a concept assumes that "a set of particular objects or a collection of concrete impressions is needed. They serve as the raw material for making a comparison, through which the common, similar, jointly held qualities of the objects are detected" (p. 6). The qualities abstracted as a result of this process of Generalization form the definition of the concept.

In the second sense, the NCTM (2000) points out that mathematics teachers should encourage the analysis and Generalization of different kinds of patterns in numerical, graphical, verbal, and algebraic terms, according to the educational level. Radford (2010) defines the Generalization of patterns as the process that consists of "grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S" (p. 42). From this process, the Generalization of the pattern or the general rule for any term of the sequence is obtained; therefore, Generalization is also considered as a product. This type of Generalization has been identified as an important process to support the development of algebraic thinking related to the study of patterns, relationships and mathematical structures (Kaput, 2008; Kieran, 2007; Kieran, Pang, Schifter, & Fong, 2016), promoting the transition from arithmetic to algebra.

Theoretical Framework: Conversational Learning

Several investigations (Alice, Heaton, & Williams, 2017; Brodie, 2013; Ditchburn, 2015; Eriksson, 2017; Huinker & Freckmanne, 2004; Jaworski, 2008; Potari, Sakonidis, Chatzigoula, & Manaridis, 2010) have shown that sharing their reflections about the teaching and learning processes promotes the broadening of the visions of the teachers about the difficulties of being a teacher and teaching. Raelin (2007) argues that reflection is interactive and that conversation among colleagues promotes it, allowing them to negotiate and develop a shared understanding of the mathematical content, its teaching and learning (Preciado-Babb, et al., 2015; Zaslavsky & Leikin, 2004).

Perspectives of Conversational Learning

Ernest (1994) argues the existence of metaphors associated with the mind and its processes underlying in various theories in the field of mathematics education. He specifically points out that the metaphor of the mind in the social constructivism is people in conversation. In fact, Vygotsky (1986) proposed that the cognitive development is fostered by processes that are first learned by means of social interaction, as is the case with conversation, to later be internalized (to become knowledge) in an individual way.

Pask (1976) made a conversational learning proposal in which communication is understood as an exchange of programs and a linguistic interaction within a generalized computational media rather than as an exchange of messages through an inert and transparent media. That is, media are active computer systems within which individuals with a mind (people and intelligent systems) converse and their feedback is inherent. So, according to Pask, conversational learning results from an interaction between two or more participants looking for an (intentional) common understanding. Figure 1 shows the conversational model proposed by Pask.

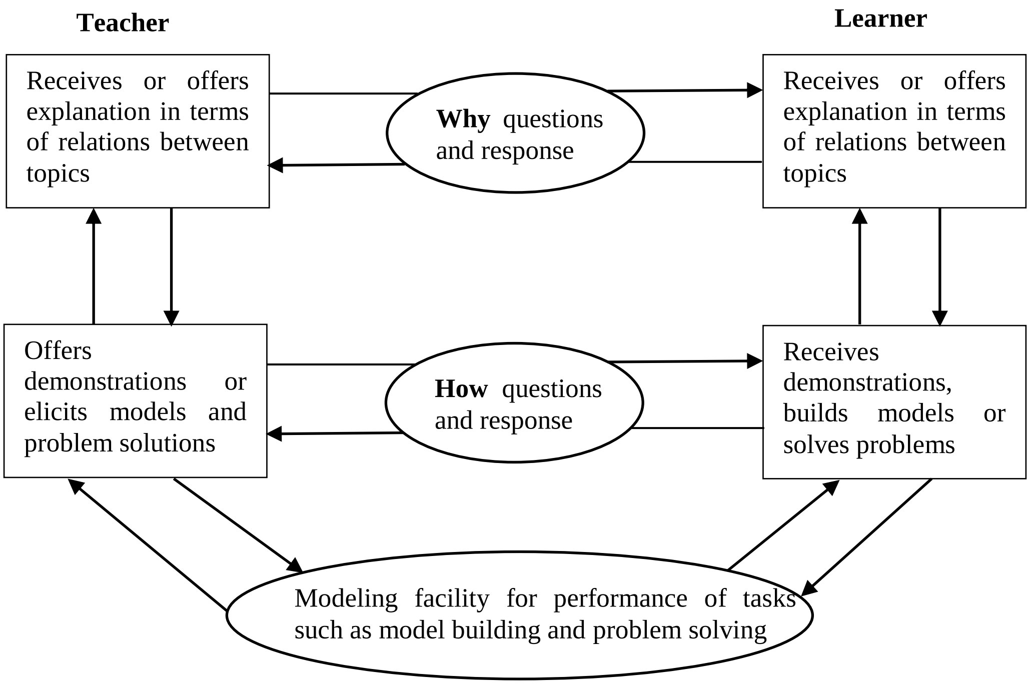

Figure 1. Model of a Conversation according to Pask (1976). Source: Scott (2001, p. 352)

The horizontal connections in Figure 1 represent verbal exchanges that can occur on at least two levels, the “how” and the “why” levels, the levels concerned with the procedure and conceptual knowledge, respectively. The vertical connections represent causal relationships with feedback (Scott, 2001). Therefore, even when the answers to a specific question on a topic are individually made, the meaning will be produced by agreements based on the conversation and learning will consist of changes in the structure of individual knowledge.

From this perspective, conversational learning is a process by which interlocutors come to know through temporal stable interpretations of their world (Ernest, 1994). Kolb and Kolb (2017) posit that conversational learning is a process by which people build new meanings and transform their collective experiences into knowledge through their conversations. That is, experience is the main source of learning and it is strengthened to the extent that people go through the cycle of experiencing, reflecting, abstracting, and acting. Figure 2 describes this cycle:

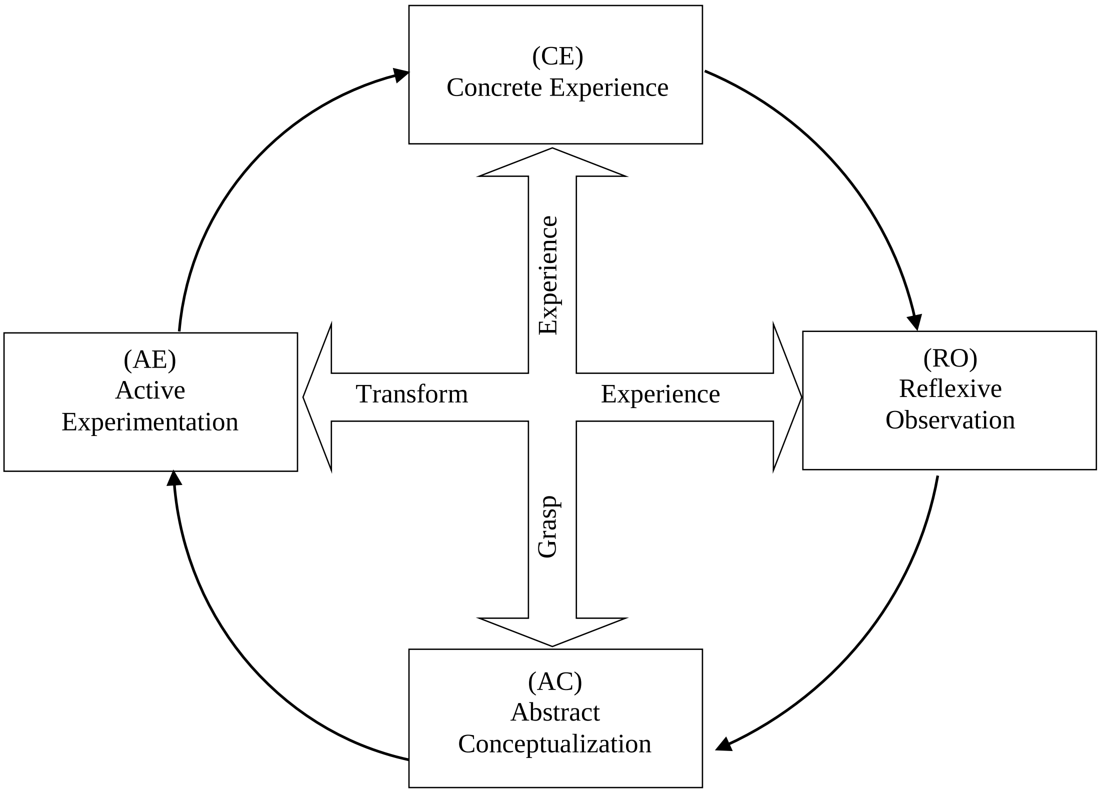

Figure 2. Experiential Learning Cycle (Kolb & Kolb, 2017, p. 32)

The central arrows in figure 2 show the dialectic relationships between different learning styles in an experiential cycle. The relationship CE – AC corresponds to how experience is attained, and the relationship RO – AE to how experience is transformed. Thus, learning in conversation implies a cyclic, dialectic, and holistic process of adaptation to the world in which transactions take place among people and the environment they are in (Dewey, 1938).

Reflective Conversation and Professional Learning

Reflection is essential to learning in conversation; however, little reference to the role of conversation is made in the above-mentioned models. Dewey (1933) said that we only learn from our experiences, when we reflect on them. Reflection captures the actions upon the individuals and gives meaning to things; consequently, it has a fundamental role in the learning process of the people (Schön, 1983). Therefore, the idea of conversation presented in the models of Pask (1976) and Kolb and Kolb (2017) extends to the idea of reflective conversation (RC) to have an understanding of how professional knowledge emerges during conversations and reflections, as well as the conditions under which this knowledge occurs.

We understand as RC, the communicative process from which the interlocutors acquire greater levels of awareness of the meanings, thoughts and actions related to a specific topic of conversation, using (implicitly or explicitly) questioning and interaction of opposites and contradictions, as referred to in the mentioned models. Figure 3 shows how the two models are integrated to investigate how RC is involved in the development of knowledge of a future mathematics teacher:

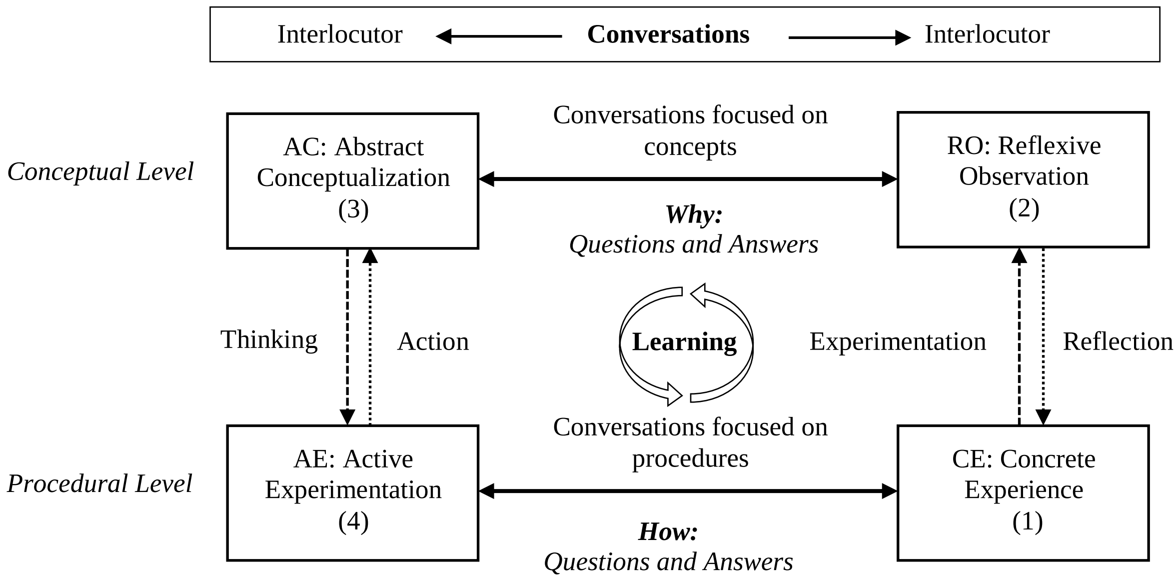

Figure 3. Reflexive Conversation and Learning based on Pask (1976) and Kolb and Kolb (2017)

Developing knowledge from an RC involves the willingness to dialogue and negotiate meanings, accept questions and argue ideas and knowledge shared, in this case, knowledge of mathematics, its teaching and learning (Bullough & Pinnegar, 2001; Earles, Parrott, & Knight, 2016). Seen in this way, the knowledge of future teachers must be related to their conversational context in such a way that a space of participation is established during the conversation to explain their ideas and knowledge related to the topic, thus enabling the emergence of meaningful relationships in that context. Since knowledge is first a social process, and then an individual one, by means of signs in general, and language in particular (Vygotsky, 1986), conversation is a tool that allows for the sharing of thoughts with others openly, and in doing so, it gives way to a reflective process that influences collective and individual meanings, thoughts and actions.

Methodology

A qualitative and interpretative approach was chosen to understand and describe how the professional knowledge of future mathematics teachers develops in the context of a reflexive conversation in the classroom.

Context and Participants in the Study

The study was carried out with ten prospective secondary and high school mathematics teachers (seven women and three men) with an average age of 22, who were enrolled in a teacher training program at a public (government financed entity) university in Mexico. The only criterion used for selecting the teachers was that they were studying a course in didactics of mathematics, and that they were in the last semester of the program. This program is four years long and its curriculum includes mathematics courses (advanced algebra, Euclidean and analytic geometry, probability, statistics, and advanced calculus), education-oriented technology and didactics of mathematics. The participation of the future teachers and their teacher in this study was voluntary, without any compensation or sanction. The teacher has a PhD in didactics of mathematics and a working experience in university education of five years and was informed about the objective and characteristics of this study.

Data Collection

The RC model was used to promote the development of knowledge about mathematical Generalization in future teachers. The teacher's participation consisted of making the ideas in the conversation flow and to promote the conversational learning cycle. An open task was posed to the future teachers as a topic of discussion considering that learning based on conversation can be initiated by interacting with questions related to a topic (Pask, 1976), and that the requirements to solve a problem are guiding factors for reflection (Dewey, 1938). The task was related to the properties of integer numbers and could be solved using arithmetic or basic algebra; it was designed in such a way that it would allow the discussion of the idea of generalization, its teaching and learning in mathematics. The instructor posed the task in the following way: *What could be said about the result of multiplying any two consecutive integer numbers?*

The task was answered in the following way: The students organized freely in teams of two or three members and provided a written answer according to their knowledge and abilities; there was no pre-established or unique strategy. The responses were then analyzed by all participants. Conversations around the analysis took place in three sessions of sixty minutes during one week. In the first session, the teacher guided the discussion towards the mathematical procedures involved in the responses to promote the CE mode; the second session focused on RO, and the third session on AC – AE. The conversations were focused on the content of the question, teaching, and learning. The future teachers decided the direction of their ideas and the order of their intervention; they did not deliberately follow the RC model. The data was obtained by non-participant observation in each session; the conversations were recorded on audio and video and were transcribed for analysis. The anonymity of the participants is preserved using the following codes: W1, W2, W3, …, W7, for women, M1, M2, M3 for men, and T for the teacher.

Data Analysis

Figure 3 shows the model used for the analysis explained in the following paragraphs:

i. The transition between the learning modes of the RC model was analyzed based on the identification of the level of conversational interaction (procedural or conceptual). The procedural level characterized conversations as discussions focused on the construction or use of mathematical procedures, while the conceptual level characterized them as focused on the use of conceptual ideas or theories to answer the posed question.
ii. The discursive expressions associated with what the future teachers experienced, reflected, thought, and made during their conversations were analyzed to identify their role in the development of professional knowledge, and
iii. How the reflections favored the modification of knowledge and constitute (new) knowledge of the group was analyzed.

Results

This research analyzed how reflective conversation among future teachers contributes to the development of their professional knowledge, specifically mathematical generalization. Four learning modes (CE, RO, AC and AE) and the transitions among them were revealed during the RC. The construction of knowledge about teaching and learning mathematical Generalization in primary education was possible through the negotiation of meanings and reflections. The results, by learning modes and sessions, are presented below. CE mode covers the first recorded conversational session, RO the second session, and AC and AE the third session.

Learning Mode CE

CE is learning by experimenting/feeling and it is placed at a procedural level. Some participants started experimenting with arithmetic cases, and others with a more general algebraic approach. The shared productions on the blackboard illustrate this (see Image 1 and 2, respectively).

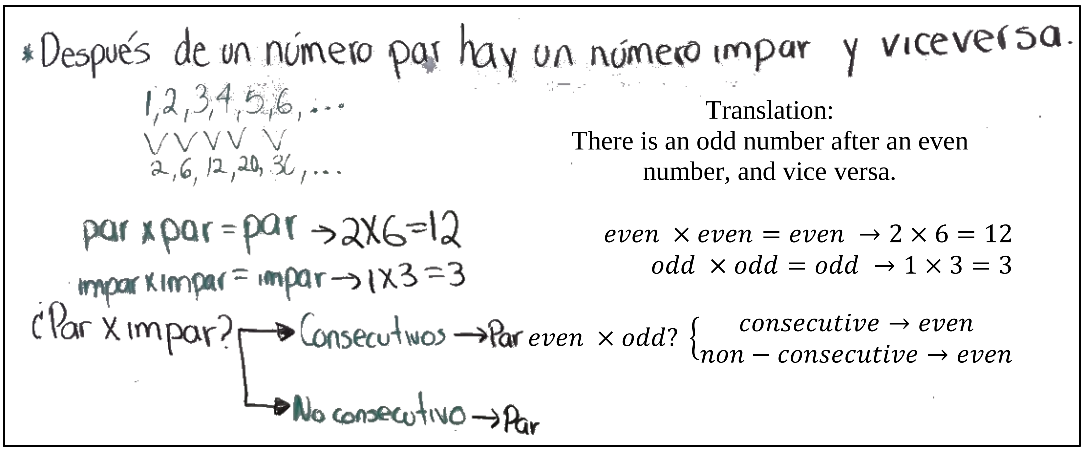

Image 1. Arithmetic Procedure Performed by W1 and W2 in CE Mode

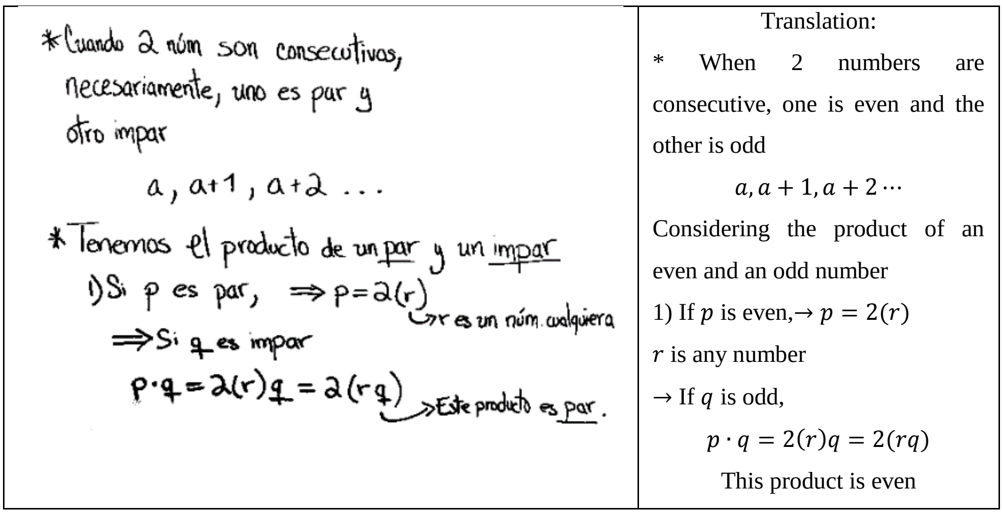

Image 2. General Algebraic Procedure Performed by W3 and M1 in CE Mode

The conversation was on a procedural level because it was based on the use of multiplicative operations with different cases of even and odd numbers, as shown in Image 1. However, they were able to identify and verify that the result of multiplying two consecutive integer numbers is an even number after sharing ideas, meanings, and procedures in the conversation. This is shown in the following excerpts:

00	T: What can be said about the result of multiplying any two consecutive integer numbers?
01	W5: (…) when the numbers are consecutive, the result is an even number, as we see here [points to the first numerical array in Image 1].
02	M2: (…), the idea in arithmetic is that a student thinks in cases, for example that $2 \times 6 = 12$ [points to Image 1]. Instead, in algebraic thinking, two consecutive numbers are considered, but bigger numbers, and then ask, what is happening? So, he is generalizing.
03	M1: (…), I considered Generalization as something out of algebraic thinking, and the identification of patterns and the relationship between quantities as something inherent to arithmetic. In this case, you can see a relationship between quantities that are growing from 2 to 4, then to 6, and then 2 units each time. As a conclusion, the result of multiplying two consecutive numbers is an even number, generalize!
04	W4: I thought that you could not generalize in arithmetic (...), but after reflecting on what we mean by generalizing, as in this case where the multiplication of two consecutive integer numbers is always an even number, then you can generalize in arithmetic and algebra. In the sequence [points to Image 1], it so happens that from 2 to 6, you need 4, and you can get that the Generalization would be multiples of two. That is, an even number of the form $2k$.
05	W2: (…) How do I interpret the expression $2k$ in arithmetic or algebra? I think that $2k$ can be seen as a whole area or an even number in arithmetic, but in algebra, it represents a quantity that is being doubled (...)

The first conversations lead to a reflection and discussion of the concept of an even number as a Generalization in arithmetic. They used $2k$ to represent it and discussed the role of Generalization in arithmetic and algebra. However, they did not achieve the property “the multiplication of two consecutive integer numbers, n and $n + 1$ is the sum of the first n consecutive even numbers” as part of the process of generalization. This is because the conversation focused on understanding the meaning of generalization, partially neglecting the task.

Learning Mode RO

RO consists of observing and reflecting (learning by processing) on what has been made/experienced in CE. In order to move to RO, the teacher questions how to relate the multiplication of two consecutive integer numbers with the idea of an even number. Then, the participants used geometrical representations to try to generalize the result looking for the general conditions needed to get an even number when multiplying two consecutive integer numbers.

The students discussed how to recognize a pattern when the product of consecutive integer numbers is increasing using the recursive calculation of differences. Thus, they obtained their first geometrical representation by exchanging ideas and reflecting on them. Image 3 shows the pattern followed by numbers 2, 6, 12, 20, and 30.

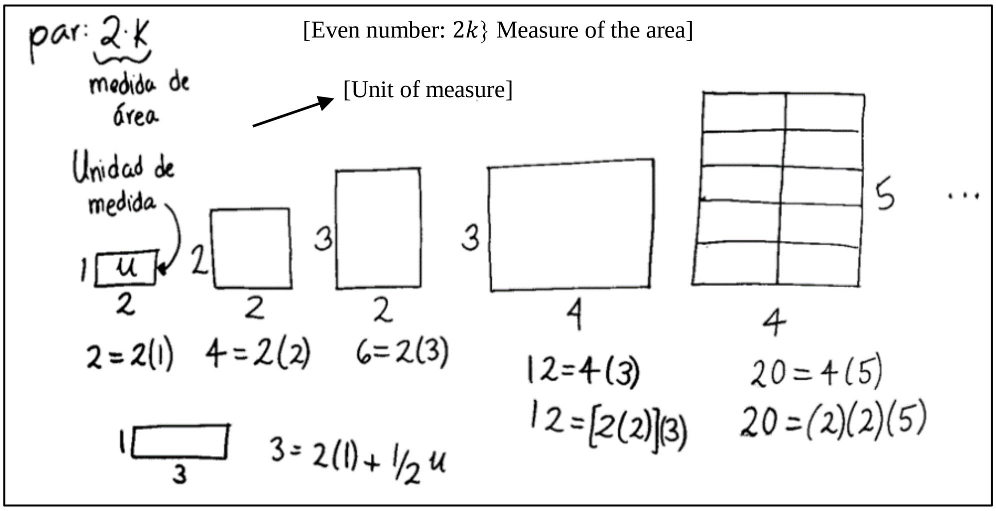

Image 3. Geometrical Representation of Even Numbers as the Measure of the Area of Rectangles given by W7

The conversations favored reflections on the importance of Generalization in mathematics and its teaching, particularly in arithmetic and basic algebra. So, the discussion moved beyond the specificity of the initial question, as shown in the following excerpts:

06	T: How could the multiplication of consecutive integers be related to the idea of an even number?
07	W5: I suggest considering the even number, given by $2k$, as the measure of the area of a rectangle of base 2 and height 1, that is, of area 2. And so on, looking for the generalization.
08	W6: But we were noticing that the multiplication of two consecutive numbers is even, no? So, how can we show that the multiplication of two consecutive numbers is really an even number?
09	W2: I think that what W5 is saying is to build or look for the behaviour in a sequence.
10	W1: There! [Image 3], it can be seen that the product of two consecutive integer numbers is even, because number 2 is the common number, which is the area in this case.
11	M1: I wonder if this representation could be used to obtain an arithmetic structure that is a Generalization [Image 3]. In fact, I think that if a primary school student knows how to generalize, then it could be said that the student also knows how to interpret.
12	W4: The concept of an even number is a Generalization made by the students, because they associate all multiples of two as an even number; may be not with those words, but they processed it in this way.
13	M2: I said something like this, it is like there is a transition towards algebraic thinking.

In this learning mode, the geometrical - visual part was essential to reflect and make ideas, thoughts and meanings of Generalization explicit; for example, understanding Generalization not only as a process, but also as the result of this process; that is, as a “process-product” relationship that must be studied in teaching mathematics, as discussed in the next paragraphs.

Learning Mode AC

AC is to theorize or generalize the experience (learning by generalizing) from RO. It was found that the participants were able to conceptualize the notion of an even number as a mathematical concept associated with the process of Generalization of multiplicative arithmetic relationships, as observed in the RO mode. But, in the AC mode, it is possible to expand and concretize when it is possible to give a symbolic expression that describes the process of generalization, as detected in the following excerpts:

14	M2: (…) Yes… and consider it as the base line for teaching even numbers.
15	M1: Yes, to build a meaning of an even number (…)
16	M3: It is interesting how we came to a more general geometrical representation (…), even when it is not very intuitive (see Image 4).
17	W7: It is the construction of a Generalization (…)
18	M1: (…) I can see another thing now and wonder, what is the meaning of the position in this context? I believe that it is a way to place the previous rectangle in the current one (see Image 4).
19	W6: The area is something common in all the rectangles of the sequence (...), the area of the initial rectangle is the unit of measure.
20	W4: m is the measure of the height and $m + 1$ is the measure of the base of the rectangles; it can be geometrically seen that the result of multiplying them (consecutive numbers) is a multiple of two, (...)
21	M1: Yes, $m(m + 1) = 2k$, in general.

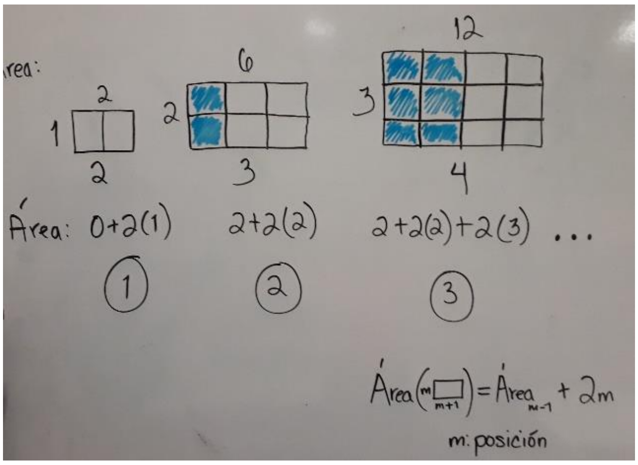

Image 4. Interpretation of an Even Number as a Mathematical Generalization given by M1

The development of thoughts was observed to the extent that the initial question reaches into broader aspects like the meanings of generalization, its teaching and learning. Thoughts and actions move from a divergent to a convergent mode. Divergent learning mode usually moves between CE and RO and convergent learning mode between AC and AE (Kolb & Kolb, 2017). This means that it moves from the generation of ideas in several directions to looking for "the best" answer to the question, looking for regularities or patterns that could lead to abstract a general concept; in this case, "the multiplication of two consecutive integer numbers gives the sum of first consecutive even numbers" as the result of a process of generalization. Unfortunately, this did not happen.

AC did not take place as hoped. A weaker property was found, even though Image 4 was suggestive:

$$1 \times 2 = 2$$

$$2 \times 3 = 2 + 4$$

$$3 \times 4 = 2 + 4 + 6$$

Etc.

Learning Mode AE

AE is to apply or test a theory for a subsequent experience (learning doing). To promote this learning mode, the teacher asked if there was any kind of arithmetic property or concept that included the concepts of even number and multiplication, as shown in the following excerpts:

22	T: Is there some kind of arithmetic property or concept that involves in some way the concepts of even number and multiplication?
23	W3: Multiplication is associated with power, maybe if we think in the square of consecutive integer numbers as we did at the beginning and see what happens (...)
24	M2: As W3 said, I think we could ask: What integer numbers result in an even number when raising them to square? Or is the square of any even number an even number?
25	W2: (…) We already know that the square of an even number is even, and that it also happens with odd numbers; but at least I had not made a geometrical interpretation as the ones made here (...).
26	W4: (…) so let's do a geometrical representation, I could think of a square of 3 units length, and then one that doubles it and see what it happens, what do you think? (W4 gives the representation shown in Image 5).
27	M1: (…) more general; I suggest considering an odd number $2m + 1$, so the subsequent number $2m + 2$ is even. Considering that $(2m + 1)^2$ is odd (as W2 said), rewriting this as: $2q + 1$ and doing the math (...) [Image 6], the result is a multiple of 2, so it is even. Furthermore, from the previous conclusion, the multiplication of an even number by an odd number is an even number, so it can be deduced that the square $(2l \times 2l)$ is even [see Image 6].
28	W1: (…) Yes! We have a geometrical explanation (…)

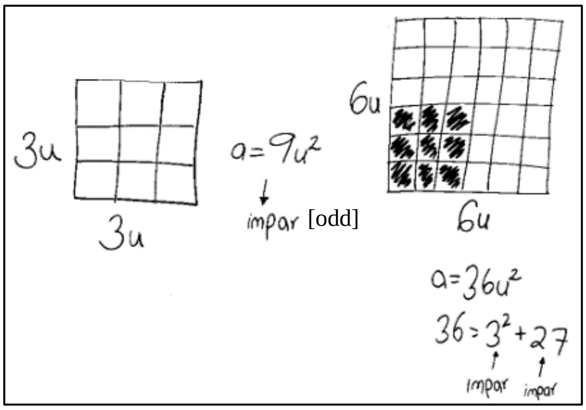

Image 5. Geometrical Representation that illustrates that the Square of an Odd Number is Odd given by W4

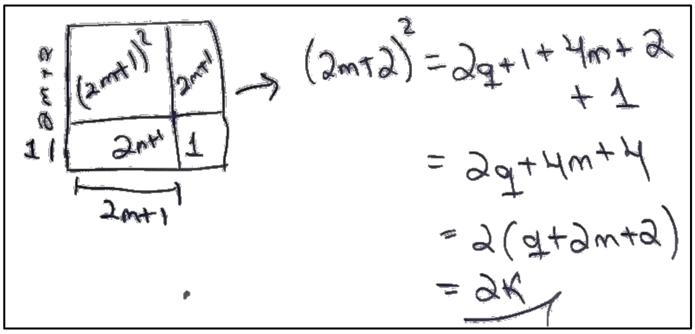

Image 6. Geometrical Representation that illustrates that the Square of an Even Number is even given by M1

Once they attained these results, the actions of the future teachers consisted of using several geometrical, arithmetic and algebraic representations to generate explanations of other types of situations that, in some way, involved a process of mathematical Generalization which corresponds to the learning mode AE. For example, they considered working with the idea that "the square of an odd number reduced by one unit, is an even number" based on different ideas and representations to get one that links and integrates them.

The actions were centered on the use of representations to obtain explanations and validate the results in a general form. Some of these representations and excerpts are shown next:

29	W5: (…) "Erase" one square unit to the representation of W4 and rearrange it in a rectangular array (see Image 7). (...) the area of the rectangular form is a multiple of 2.
30	W1: yes, it's true! These geometrical images (see Image 7) illustrate that the even number also comes from a process like this one (…).
31	W6: (…), It seems that we should always rearrange to make things clearer (…)
32	M2: (…), a more general way could be looking for a numerical pattern to see what is happening or where we go, and from these particular cases identify that the decomposition in factors keeps 2^3 invariant (see Image 8).
33	M3: (…) from the last session we know that $n(n+1)$ is even, then developing and organizing the expression given by M2 (see Image 8), we get that the square of an odd number reduced by one, is even (see Image 9).

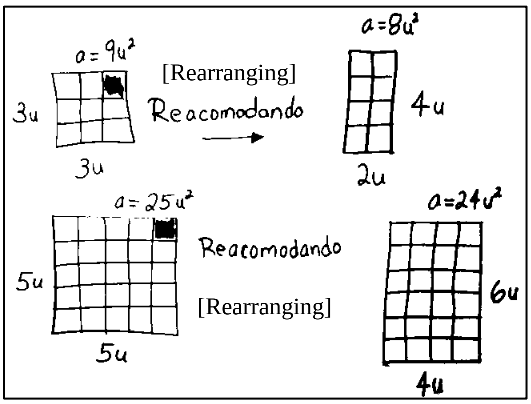

Image 7. Geometrical Representation of the Square of an Odd Number Reduced by one, given by W5

$$8 = 2 \cdot 2 \cdot 2 = 2^3$$
$$24 = 12 \cdot 2 = 2^3 \cdot 3$$
$$48 = 6 \cdot 2^3$$

Image 8. Numerical Development of the Square of an Odd Number Reduced by one, given by M2

$$\frac{(2n+1)^2-1}{8} = \frac{4n^2+4n+1-1}{8} = \frac{4(n^2+n)}{8} = \frac{n^2+n}{2} = \frac{n(n+1)}{2} = p \qquad p \in \mathbb{Z}$$

Image 9. Algebraic Development of the Square of an Odd Number Reduced by one, given by M3

At the end of the discussions, the participants agreed that it is possible to obtain a Generalization that explains the following result "the square of an odd number reduced by one is an even number" with the process shown in Image 9, because this is a more abstract process if there is no arithmetic and geometric support.

In summary, the participants applied what they had learned (arithmetic and geometric concepts and procedures) in a new experience, where they shared thoughts and actions, questioned, and discussed them in an open and flexible environment. Therefore, they learned from experimenting with new situations. It is worth noting that, for this mode of learning, the participants did not properly apply what they learned in AC; they used instead the "abstracted" property (as observed in the excerpt of M3 in the dialogue, line 33) to validate and explain a new property by means of a process of Generalization and the use of geometric, arithmetic, and algebraic representations.

Reflective Conversation and the Development of Professional Knowledge

Conversations and reflections guided by questionings, the negotiation of ideas and meanings led future teachers to the development of professional knowledge of mathematical (arithmetic-algebraic), and some didactic implications for its teaching and learning in basic education. For example, the importance of using several representations to support the transition from arithmetic to algebraic thinking, as well as formulating general procedures and outcomes.

This is how three reflection modes related to the mathematical and didactic content of the topics discussed were identified: *Anticipated reflections*: these are characterized by references to future actions; *updated reflections*: they are expressed in present indicative tense; and *retrospective reflections*: they are expressed in past tense. Anticipated reflections and retrospective reflections were made in relation to teaching and learning the arithmetic. Updated reflection was made about the arithmetical content. Some excerpts related to these reflections are shown here.

Anticipated Reflections

34	M2: (…) we should ask if this is trivial for a primary school child. Yes? No? Why? And then, if it is trivial for high school or university students (...) Could a primary school child solve it?
35	M1: (…) moving from arithmetic to algebraic thinking. I don't know if recognising an algebraic or arithmetic structure, means that we are already in that transit. It is not usually like this when we are in primary or secondary school, (...) In primary school the data is particularised, or not? That is, it is said that an even number multiplied by an even number is even, or that odd by odd is odd, but the doubt always remains (...)
36	W4: (…) the concept of an even number is a Generalization made by the students, because they associate all multiples of 2 as an even number, (...), they process it in this way (...)
37	M2: (…) I said something like this, it is as if there is a transition to algebraic thinking.
38	M1: (…) if a primary school student knows how to generalize, then he also knows how to interpret (…)
39	M2: The activity could be used to teach even numbers… it is like building the meaning of even.
40	W4: I should start with the first method proposed [Image 1] because it is primary school, that they see something particular, the differences, and then the relationship between quantities. And reach a generalization.
41	M3: How can something more general be achieved?…
42	M2: Something more general could be looking for a numerical pattern to see what is happening or where can we go (…) from particular cases…
43	M1: Yes, a Generalization could be achieved with M1's explanation [algebraic representation of even and odd numbers] but the articulation with the geometric part allows understanding why it is true.
44	M2: I also consider it would be useful to articulate the representations, because M1's explanation is more abstract, it does not have the geometrical support. (…) it could be use as an argument for the students.

Retrospective Reflections

45	M1: (…), I considered Generalization as something out of the algebraic thinking, and the identification of patterns and the relationship between quantities as something inherent to arithmetic. In this case, you can see a relationship between quantities that goes from 2 to 4, and then 2 units each time (…)
46	W4: (…) I thought that you could not generalize in arithmetic (...), but after reflecting on what we mean by generalizing, (...) then you can generalize in arithmetic and algebra (...)
47	M1: (…) I am seeing it now and I wonder: what does the position mean in this context? I believe that it is one way to place the previous rectangle in the actual one (...)
48	W2: (…) I am realizing that we didn't come to a generalization, what we did was a geometric explanation (…)

Updated Reflections

49	M2: (…), the idea in arithmetic is that the student thinks in cases, for example, that $2 \times 6 = 12$. (...) in algebraic thinking, two consecutive numbers are considered, but bigger ones, and asked: what is happening? So, he is generalizing (...).
50	W2: (…) How do I interpret the expression $2k$ in arithmetic and algebra? (...) $2k$ can be seen as the whole area (...) in arithmetic, but in algebra, it represents a quantity that is being doubled (…).
51	M1: (…) I wonder if this representation (see Image 3) could be used to obtain an arithmetic structure that is a Generalization (...)
52	W6: Something common to all rectangles of the sequence is the number, which is the area in this case (…)
53	W1: It is true! You can see with these geometric images that the result of this process is also an even number (…)

Questions and contradictions surrounding the procedures and concepts used to answer the questions posed, promoted conversations focused on the negotiation of ideas and meanings, specifically, on the relationships between the concepts of an even number and Generalization in mathematics, and its implications for its teaching, as shown in the following excerpts:

54	W4: (…) I proposed that there could be no Generalization in arithmetic, because I had an idea (...), but after reflecting on what we mean by Generalization (...), I realise that there can be Generalization in arithmetic and algebra, but in different ways (…)
55	W5: (…), Mathematics is about patterns, relationships, and generalization, (...), then, (...) generalizations (...) appear in all areas of mathematics (...).
56	M2: (…) we should ask if this is trivial for a primary school child. Yes? No? Why? (…)
57	M1: (..) if a primary school student knows how to generalize, then (...) he also knows how to interpret.
58	W4: (…) an even number is a Generalization made by the students (...) they process it in this way.
59	M2: (…) there is a transition to algebraic thinking.
60	M2: (…) it could be thought as the base line for teaching even numbers.
61	M1: (…) to build a meaning for even numbers (…)
62	W1: (…) These geometrical images illustrate that the even number also comes from a process (…)
63	W6: (…) it seems that we should always rearrange to make things clearer (…)
64	W2: (…) how do I interpret the expression $2k$ in arithmetic and algebra?
65	W6: (…) it depends on the analysis of $2k$. If we consider k as a general number, and not necessarily as an increasing variable; I should say that it depends on the approach (…)
66	W8: (…) ok, but (…) the arithmetic and algebraic interpretation of $2k$ is not the same as in calculus. In arithmetic, it can be seen as the whole area or the representation of an even number; but in algebra I interpret it as a change or variation, the quantity is being doubled (…)
67	M2: (…) If I am in Calculus, it could represent a lineal behaviour of a function (…)
68	M1: So, what is the conclusion, is $2k$ an algebraic or arithmetic structure? (…)

In addition to becoming more aware of their actions, thoughts and meanings, the future teachers also relate them to their implications for the teaching of arithmetic and algebra. For example, with the ability to move from arithmetic to algebraic thinking (see excerpts 55 – 60 and 64 – 68). Figure 4 synthesizes the process of the professional knowledge development by the future teachers during the reflective conversation.

<table>
<tr><td colspan="2">Procedural level: Procedures based on the use of representations and Generalization as a process.</td><td colspan="2" rowspan="2">Knowledge:
Mathematic - Didactic</td><td colspan="2">Conceptual level: Concepts of an even number, power, and mathematical Generalization (as a concept).</td></tr>
<tr><td>(1) Mode CE</td><td>Example</td><td>(2) Mode RO</td><td>Example</td></tr>
<tr><td>Use of arrays and analysis of numerical cases</td><td>Image 1</td><td rowspan="4">Mathematical concepts such as an even number require the construction of arithmetic-algebraic-graphic representations for its understanding and teaching in basic education.</td><td rowspan="4">Mathematical Generalization is a method and an ability that should be promoted in basic education to interpret, validate, and explain mathematical results.</td><td>Use of general algebraic structures</td><td>Image 2</td></tr>
<tr><td colspan="2">Transformation of knowledge</td><td colspan="2">Comprehension of knowledge</td></tr>
<tr><td>(4) Mode AE</td><td>Example</td><td>(3) Mode AC</td><td>Example</td></tr>
<tr><td>Use Generalization and representations to validate and explain new results: $[(2n+1)^2-1]$ $n \in Z$, is an even number.</td><td>Image 9</td><td>Patterns are established and an even number is conceptualized as a Generalization of the multiplication of consecutive integers.</td><td>Image 4</td></tr>
</table>

Figure 4. Transition between Modes of Learning and Professional Knowledge of Future Teachers in RC

Discussion

This study verified that RC promotes the development of knowledge among future mathematics teachers by questioning the nature and construction of the concept of an even number from a mathematical and didactic point of view, and the search for answers freely and openly. The participants revealed different modes of learning and types of thinking about the representations of even numbers for their teaching and the implications for students' learning when discussing the properties of even numbers. In this sense, Toom et al. (2015) report that pre-service teachers move among different modes of learning. We found that prospective teachers move among four learning modes, for example, from the mode of concrete experimentation (CE), to reflexive observation (RO), to abstract conceptualization (AC), and to active experimentation (AE). RC encouraged the transition among these learning modes based on posing questions and answers regarding the structure of an even number and the role of Generalization for its teaching at both the procedural and conceptual levels. Anticipated, updated and retrospective reflections on the concept of an even number, its properties and teaching produced reinterpretations and expansion of the theoretical knowledge of the topics discussed (even numbers and generalization) of future teachers, connecting them to actions of their professional practice. In this way, the future teachers developed professional knowledge associated with mathematical Generalization based on the RC of the mathematical content and its didactic implications.

The knowledge developed consisted of conceiving Generalization as an important process to explore, explain and validate mathematical results and recognising its importance as a mathematical ability to be developed during the teaching of arithmetic, geometry, and algebra in basic education. In addition, it was realised that many mathematical concepts are the product of generalizations, as in the case of the concept of even number. Moreover, the future teachers generated reflections on the teaching of even number and the process of Generalization with emphasis on pattern recognition and argumentation of mathematical results. In particular, the importance of using several representations (arithmetic, algebraic, and geometric) to generalize and prove mathematical results in the teaching of mathematics was recognised. Articulating various semiotic representations is an essential pedagogical knowledge for professional practice, as it helps the understanding of mathematical objects (Iori, 2017).

The development of knowledge was enhanced by the dialogue guided by the teacher, as it allowed sharing questions and answers, attending to contradictions, and accepting different modes of thinking in an atmosphere of trust and total respect of ideas. This agrees with Bakhtin (1986), who mentioned that dialogue leads to discussion and meanings, and it is in the same direction as the study of Simoncini, Lasen and Rocco (2014), which proposed that a guided dialogue empowers future teachers to obtain better perspectives of their teaching practices, including their thoughts and actions, as could be seen in this study. The results are consistent with the reports of Demissie (2015), Jaworski (2008), and Chamoso, Cáceres and Azcárate (2012) that showed that participating in collective inquiry promotes reflective thinking among peers. As mentioned, questioning about mathematical content with emphasis on conceptual relationships and the rationale of the solutions over the procedures made possible the emergence of several types of reflection and the modification in modes of thinking, of both mathematical content and its teaching and learning. The empirical evidence collected confirms the relationship between reflective conversation and professional learning described in Figure 3.

Conclusion

Reflective conversation contributes to the learning and development of professional knowledge to the extent that questions are posed collectively, and the answers provided are not only focused on how the mathematical solution to a task is obtained, but the rationale of this solution. It can also be obtained if the connection between the theoretical and practical knowledge is promoted during the RC, for example, by sharing experiences, reflections, actions, and thoughts. Thus, we believe that the creation of spaces for reflective conversation characterized by open and genuine discussion among future teachers may contribute to the generation of collective reflection and learning processes regarding the future professional practice. This broadens the idea that reflection is an essential aspect for the learning of teachers; however, it has been documented that its efficient development requires scaffolding for its integration with the professional practice (Roberts, 2016).

Generating spaces for RC could be part of that scaffolding. In this sense, this study identifies two meanings for RC associated with the learning of teachers: the first one, RC as a means for social construction of professional knowledge, and the second one, as a tool to promote the personal professional development based on experience. Finally, it is necessary that training programs enable future teachers to transform their initial perception of teaching mathematics and become aware of the mathematical and pedagogical knowledge needed for their professional practice (Ponte & Chapman, 2016). Therefore, we plan to investigate how to incorporate the RC in teacher training programs, so that, as it happened in this study, it becomes useful in the development of professional learning and to promote anticipated reflections regarding the professional practice, and even to encourage a shared vision and commitment to the practice (Preciado-Babb et al., 2015; Toom, Husu, & Patrikainen, 2015).

References

Alice, M., Heaton, R., & Williams, M. (2017). Translating Professional Development for Teachers into Professional Development for Instructional Leaders. *Mathematics teacher educator, 6*(1), 27 – 39. http://dx.doi.org/10.5951/mathteaceduc.6.1.0027.

Arcavi, A. (2016). Promoviendo conversaciones entre docentes acerca de clases filmadas de Matemáticas. *Cuadernos de Investigación y Formación en Educación Matemática, 11*(15), 385 – 396.

Bakhtin, M. M. (1986). *Speech genres and other late essays*. Austin, Texas: University of Texas Press

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389 – 407. https://doi.org/10.1177/0022487108324554.

Brodie, K. (2013). The power of professional learning communities. *Education as Change, 17*(1), 5 – 18. http://dx.doi.org/10.1080/16823206.2013.773929.

Bullough, R., & Pinnegar, S. (2001). Guidelines for quality in autobiographical forms of self-study research. *Educational Researcher, 30*(3), 13 – 21. https://doi.org/10.3102/0013189X030003013.

Campbell, M., & Stohl, H. (2017). Examining Secondary Mathematics Teachers' Opportunities to Develop Mathematically in Professional Learning Communities. *School Science and Mathematics, 117* (3 – 4), 115 – 126. https://doi.org./10.1111/ssm.12209.

Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, A., Ribeiro, M. y Muñoz-Catalán, M. C. (2018). The mathematics teacher's specialized knowledge (MTSK) model. *Research in Mathematics Education, 20*(3), 236 – 253, DOI: 10.1080/14794802.2018.1479981.

Chamoso, J. M., Cáceres, M. J., & Azcárate, P. (2012). Reflection on the teaching-learning process in the initial training of teachers. Characterization of the issues on which pre-service mathematics teachers reflect. *Teaching and Teacher Education, 28*(2), 154 – 164. https://doi.org/10.1016/j.tate.2011.08.003.

Darling-Hammond, L., Chung, R., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional Learning in the Learning Profession: A Status Report on Teacher Development in the United States and Abroad.* Oxford: National Staff Development Council.

Davydov, V. (1990). Type of generalization in instruction: Logical and psychological problems in the structuring of school curricula. In J. Kilpatrick (Ed.), *Soviet studies in mathematics education* (Vol. 2).

Reston, VA: National Council of Teachers of Mathematics.

Demissie, F. (2015). Promoting Student Teachers' Reflective Thinking Through a Philosophical Community of Enquiry Approach. *Australian Journal of Teacher Education, 40*(12), 1 – 13.

Demonty, I., Vlassis, J., & Fagnant, A. (2018). Algebraic thinking, pattern activities and knowledge for teaching at the transition between primary and secondary school. *Educational Studies in Mathematics, 99*(1), 1 – 19. https://doi.org/10.1007/s10649-018-9820-9.

Dewey, J. (1933). *How we think*. New York: Dover Publications.

Dewey, J. (1938). *Education and Experience*. New York, NY: Horace Liveright

Ditchburn, G. M. (2015). Remembering Reflection in Pre-Service Teachers' Professional Experience. *Australian Journal of Teacher Education, 40*(2), 94 – 111. http://dx.doi.org/10.14221/ajte.2015v40n2.7.

Dörfler, W. (1991). Forms and means of generalization in mathematics. In A. J. Bishop, S. Mellin-Olsen, & J. Van Dormolen (Eds.), *Mathematical Knowledge: Its growth through teaching* (pp. 61 – 85). Dordrecht: Springer.

Eriksson, A. (2017). Pre-service teachers' questions about the profession during mentoring group conversations. *European Journal of Teacher Education, 40*(1), 76 – 90.

Ernest, P. (1994). Conversation as a Metaphor for Mathematics and Learning. *Proceedings of BSRLM Day Conference* (pp. 58 – 63). Manchester, England: Manchester Metropolitan University.

Horn, I. S., & Little, J. W. (2017). Attending to Problems of Practice: Routines and Resources for Professional Learning in Teachers' Workplace Interactions. *American Educational Research Journal, 47(1)*, 181 – 217. https://doi.org/10.3102/0002831209345158.

Hord, S. (2008). Evolution of the professional learning community: Revolutionary concept is based on intentional collegial learning. *Journal of Staff Development, 29*(3), 10 – 13.

Huinker, D., & Freckman, J. (2004). Focusing Conversations to Promote Teacher Thinking. *Teaching Children Mathematics, 10*(7), 352 – 357.

Iori, M. (2017). Objects, signs, and representations in the semio-cognitive analysis of the processes involved in teaching and learning mathematics: A Duvalian perspective. *Educational Studies in Mathematics, 94*, 275-291. https://doi.org/10.1007/s10649-016-9726-3

Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education, 9*, 187 – 211.

Jaworski, B. (2008). Mathematics teacher educator learning and development. In Jaworski, B. and Wood, T. (Eds.), *The International Handbook of Mathematics Teacher Education, Volume 4: The Mathematics Teacher Educator as a Developing Professional*. Rotterdam: Sense Publishers.

Kaminski, E. (2003). Promoting pre-service teacher education students' reflective practice in mathematics. *Asia-Pacific Journal of Teacher Education*, *31*(1), 21 – 32. http://dx.doi.org/10.1080/13598660301619.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5 – 17). New York: Routledge.

Kieran, C., Pang, J., Schifter, D. & Fong, S. (2016). *Early Algebra. Research into its nature, its learning, its teaching*. Hamburg: Springer Open.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester (Ed.), *Second handbook of research on mathematics* *teaching and learning* (Vol. 2, pp. 707 – 762). Greenwich, CT: Information Age Publishing.

Kolb, A. & Kolb, D. (2017). *The Experiential Educator. Principles and Practices of Experiential Learning*. Kaunakakai, Hawaii: EBLS Press.

Krainer, K., & Llinares, S. (2010). Mathematics Teacher Education. In P. Peterson, E. Baker, & B. McGaw (Eds.), *International encyclopedia of education* (Vol. 7, 702 – 705). Oxford, Uk: Elsevier.

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM: The International Journal on Mathematics Education, 40*(5), 873 – 892. https://doi.org/10.1007/s11858-008-0141-9.

Kunter, M., Baumert, J., Blum, W., Klusmann, U., Krauss, S., & Neubrand, M. (Eds). (2013). *Cognitive activation in the mathematics classroom and professional competence of teachers. Results from the COACTIV project.* New York: Springer.

Llinares, S. (2012). Construcción de conocimiento y desarrollo de una mirada profesional para la práctica de enseñar matemáticas en entornos en línea. *Avances de Investigación en Educación Matemática, 2,* 53 – 70.

Moss, J. & London McNab, S. (2011). An Approach to Geometric and Numeric Patterning that Fosters Second Grade Students' Reasoning and Generalizing about Functions and Co-variation. In: Cai J., Knuth E. (Eds.) *Early Algebraization. Advances in Mathematics Education* (pp. 277 – 301). Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-17735-4_16.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Pask, G. (1976). *Conversation theory. Applications in education and epistemology.* New York: Elsevier Scientific Publishing Company.

Pino-Fan, L., Godino, J. D., & Font, V. (2018). Assessing key epistemic features of didactic-mathematical knowledge of prospective teachers: the case of the derivative. *Journal of Mathematics Teacher Education, 21*(1), 63 – 94. http://dx.doi.org/10.1007/s10857-016-9349-8.

Ponte, J. P. (2012). Estudiando el conocimiento y el desarrollo profesional del profesorado de matemáticas. En N. Planas (Ed.), *Teoría, crítica y práctica de la educación matemática (*pp. 83 – 98). Barcelena: Graó.

Ponte, J. P., & Chapman, O. (2008). Preservice mathematics teachers' knowledge and development. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 225 – 263). New York, NY: Routldge.

Ponte, J. P., & Chapman, O. (2016). Prospective Mathematics Teachers' Learning and Knowledge for Teaching. In L. English & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (3rd ed., pp. 275 – 296). New York, NY: Routldge.

Potari, D., Sakonidis, H., Chatzigoula, R., & Manaridis, A. (2010). Teachers' and researchers' collaboration in analysing mathematics teaching: A context for professional reflection and development. *Journal of Mathematics Teacher Education, 13*(6), 473 – 485. https://doi.org/10.1007/s10857-010-9161-9.

Preciado-Babb, A., Metz, M., & Marcotte, C. (2015). Awareness as an Enactivist Framework for the Mathematical Learning of Teachers, Mentors and Institutions. *ZDM: The International Journal on Mathematics Education, 47*(2), 257 – 268. https://doi.org/10.1007/s11858-014-0657-0.

Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education*

Research Journal, 26(2), 257 – 277. https://doi.org/10.1007/s13394-013-0087-2.

Radford, L. (2010). Layers of generality and types of generalization in pattern activities. *PNA: Revista de Investigación en Didáctica de la Matemática, 4*(2), 37 – 62.

Raelin, J. A. (2007). Toward an epistemology of practice. *Academy of Management Learning & Education, 6*(4), 495 – 519.

Rasmussen, K. (2016). Lesson study in prospective mathematics teacher education: didactic and paradidactic technology in the post-lesson reflection. *Journal of Mathematics Teacher Education, 19*(4), 301 – 324.

Roberts, P. (2016). Reflection: A Renewed and Practical Focus for an Existing Problem in Teacher Education. *Australian Journal of Teacher Education, 41*(7), 19 – 35. http://dx.doi.org/10.14221/ajte.2016v41n7.2.

Rowland, T., & Ruthven, K. (2011). Introduction: mathematical knowledge in teaching. In T. Rowland, & K. Ruthven (Eds.), *Mathematical Knowledge in teaching* (1 – 5). Dordrecht: Springer.

Santagata, R., & Guarino, J. (2011). Using video to teach future teachers to learn from teaching. *ZDM Mathematics Education, 43,* 133 – 145. https://doi.org/10.1007/s11858-010-0292-3.

Saylor, L. L., & Johnson, C. C. (2014). The Role of Reflection in Elementary Mathematics and Science Teachers' Training and Development: A Meta Synthesis. *School Science and Mathematics, 114*(1), 30 – 39. https://doi.org/10.1111/ssm.12049.

Schön, D. (1983). *The reflective practitioner: How professionals think in action*. London: Temple Smith.

Scott, B. (2001). Gordon Pask's conversation theory: A domain independent constructivist model of human knowing. *Foundations of Science, 6*(4), 343 – 360.

Schoenfeld, A., & Kilpatrick, J. (2008). Towards a theory of proficiency in teaching mathematics. In D. Tirosh, & T. L. Wood (Eds.), *Tools and processes in mathematics teacher education* (pp. 321 – 354). Rotterdam: Sense Publishers.

Shoenfeld, A. (2011). *How we think: a theory of goal-oriented decision making and its educational applications.* New York: Routledge.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4 – 14.

Shulman, L. S. (2005). *The Signature Pedagogies of the Professions of Law, Medicine, Engineering, and the Clergy: Potential Lessons for the Education of Teachers*. Delivered at the Math Science Partnerships (MSP) Workshop: "Teacher Education for Effective Teaching and Learning" Hosted by the National Research Council's Center for Education, February 6 – 8, Irvine, CA.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11,* 499 – 511.

Simoncini, K. M., Lasen, M., & Rocco, S. (2014). Professional dialogue, reflective practice and teacher research: Engaging early childhood pre-service teachers in collegial dialogue about curriculum innovation. *Australian Journal of Teacher Education, 39*(1), 27 – 44.

Smith, C. R. (2015). *Continuous professional learning community of mathematics teachers in the western cape: developing a professional learning community through a school-university partnership*. Unpublished doctoral dissertation, University of the Western Cape, Bellville, South Africa.

Szydlik, J. E., Parrott, A., & Belnap, J. K. (2016). Conversations to Transform Geometry Class. *Mathematics Teacher, 109*(7), 507 – 513. http://dx.doi.org/10.5951/mathteacher.109.7.0507.

Thanheiser, E., Browning, C., Edson, A., Lo, J., Whitacre, I., Olanoff, D. & Morton, C. (2014). Prospective Elementary Mathematics Teacher Content Knowledge: What Do We Know, What Do We Not Know, and Where Do We Go? *The Mathematics Enthusiast, 11*(2), 433 – 448.

Toom, A., Husu, J., & Patrikainen, S. (2015). Student teachers' patterns of reflection in the context of teaching practice. *European Journal of Teacher Education*, *38*(3), 320 – 340.

Vygotsky, L. S. (1986). *Thought and language*. Cambridge, MA: MIT Press.

Warren, E., Trigueros, M., & Ursini, S. (2016). Research on the learning and teaching of algebra. In A. Gutiérrez, G. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education. The journey continues* (pp. 73-108). Rotterdam: Sense Publishers.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational studies in mathematics, 49*(3), 379 – 402.

Zaslavsky, O., & Leikin, R. (2004). Professional development of mathematics teacher educators: Growth through practice. *Journal of mathematics teacher education, 7*(1), 5 – 32.

Zaslavsky, O. (2008). Meeting the challenges of mathematics teacher education through design and use of task that facilitate teacher learning. In B. Jaworski & T. Wood (Eds.), *The Mathematics Teacher Educator as a Developing Professional. Handbook of Mathematics Teacher Education* (Vol. 4, pp. 93 – 114). Rotterdam, The Netherlands: Sense Publishers.

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