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Preservice Mathematics Teachers' Determination Skills of Proof Techniques: The Case of Integers

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Abstract

The aim of the study is to reveal the preservice mathematics teachers' ability to determine the techniques of proofs on integers. A qualitative case study approach was adopted in this study. The participants of the study consisted of 172 preservice teachers enrolled in an elementary mathematics teaching program in their second and third years at a state university in Turkey. The data of the study were obtained from the Proof Techniques Determination Form (PTDF) which consists of six proofs on integers proven by different techniques and semi-structured interviews with five preservice teachers who were successful in different achievement levels of PTDF. The preservice teachers were asked to determine the proof techniques presented to them and express their warrants. The proof techniques used in PTDF are; direct proof, proof by induction, proof by contradiction, proof by contraposition, proof by counterexample and proof by confliction. At the end of the study, the preservice teachers were successful in determining proof by induction and direct proof technique. However, they were unsuccessful in determining proof by contraposition and proof by contradiction technique. While there was no difficulty in determining proof by counterexample technique, most of the preservice teachers had struggled in determining proof by confliction used to show that the proposition was false. The preservice teachers mostly used direct proof instead of proof by contraposition, proof by contraposition instead of proof by contradiction, and direct proof instead of proof by confliction. It was determined that the preservice teachers tended to evaluate the technique of any proof as direct proof. Sources of these difficulties were: deficiencies in understanding the differences between proof by contradiction and proof by contraposition, usage of same warrants of both techniques, and accepting general warrants which are valid for every proof as the property of direct proof.

Introduction

The meaning of proof is to reveal the real aspect of something by showing evidence (Turkish Language Society, 2015). Various mathematics education researchers have tried to describe mathematical proof by emphasizing its different aspects. According to Yıldırım (2014), proof is an effort to accept the truth or falsity of a judicial assertion or outcome by showing adequate evidence. Almeida (2003) considered that mathematical proof is used to verify a result, communicate and persuade others to accept this conclusion, explore a conclusion, and place the results into a deductive system. By acting on the common characteristics of the definitions of mathematical proof in the mathematics education literature, it can be considered that mathematical proof is mathematically and logically general, accurate and persuasive arguments, constructed to show the truth or falsity of a proposition, consisting of known mathematical elements (definition, theorem and axioms), progressing in an axiomatic structure based on hypotheses (Doruk and Kaplan, 2017).

According to Baki (2014), the purpose of mathematical proof is to show the truth or falsity of a claim. Ko (2010) stated that the primary goal of proof and counterexamples in mathematics was to demonstrate the truth or falsity of a proposition. According to Ko and Knuth (2009), justifying and falsifying are important skills in advanced mathematical thinking as they help to show whether the propositions are true or false. Lakatos (1976) emphasized the importance of falsifying in mathematics, explaining that it is also important to show the falsity of a proposition as well as to show its truth, which has an important place in the development of mathematics.

Four basic techniques often used to prove the truth of mathematical propositions are; direct proof, proof by induction, proof by contraposition, and proof by contraposition (Akkas et al., 1998; Callıalp, 1999; MEB [Ministry of Education], 2013). In direct proof, one attempts to demonstrate $p \Rightarrow q$ directly. It is asusumed that p is true and p is used to show that q must be true. Proof by contradiction is grounded in the fact that any proposition must be either true or false, but not both true and false at the same time. This technique can be used to demonstrate $p \Rightarrow q$ by assuming $(p \Rightarrow q)' \equiv p \wedge q'$ are simultaneously true and deriving a contradiction. Proof by contraposition is a proof technique but not a method per se; it is just the implication that $p \Rightarrow q$ is equivalent to $q' \Rightarrow p'$. The second proposition is the contraposition of the first proposition. By stating that the two propositions are equivalent we mean that if one can prove $p \Rightarrow q$, then they have also proven $q' \Rightarrow p'$. Proof by induction is a very powerful technique which uses recursion to demonstrate an infinite number of facts in a finite amount of space. This technique consists of three steps: showing that a propositional form *P(x)* is true for some basis case, assuming that *P(n)* is true for some *n*, and showing that this implies that *P(n+1)* is true. Then, by the principle of induction, the propositional form *P(x)* is true for all n greater or equal to the basis case.

In order to illustrate the falsity of mathematical propositions, the techniques of counterexample and confliction are used (Akkas et al., 1998). The falsity of mathematical propositions is often supported by counterexamples (Altun, 2014; Lampert, 1990; Yasuhiro, 1991). While a mathematical proof shows the accuracy of the proposition for all cases (Stylianides and Stylianides, 2009), a counterexample indicates that the present premise is incorrect (Akkas et al., 1998; Irmak, 2008). In proof by counterexample $(p \Rightarrow q)'$ is logically equal to $p \wedge q'$. If an example is found indicating that $p \wedge q'$ is true, this indicates that proposition $p \Rightarrow q$ is false. If it is not known whether a proposition is true or false in the proof techniques using confliction, this proposition is assumed to be true, and some results are obtained without a proposition. If the results obtained contradict with each other or with a known element, it is concluded that the proposition is false (Akkaş et al., 1998).

Earlier studies showed that undergraduate students and mathematics teachers had difficulty in constructing proof (Cusi and Malara, 2007; Doruk and Kaplan, 2015; Ko and Knuth, 2009; Weber, 2001), generating a counterexample (Riley, 2003; Zaslavsky and Peled, 1996), and evaluating proofs constructed by others (Alcock and Weber, 2005; Doruk and Kaplan, 2013; Knuth, 2002; Martin and Harel, 1989; Morris, 2002; Segal, 2000; Selden and Selden, 2003; Uygan, Tanışlı and Köse, 2014). These studies reported that students had difficulties in interpreting the mathematical definitions, comprehending concepts, understanding the theorem, using mathematical language and notation, choosing the appropriate proof technique, constructing proof by using definitions, expressing their thinking, and selecting an element from the domain at the beginning of proof (Doruk and Kaplan, 2015; Güler, 2013; Moore, 1994, Selden and Selden, 2003; Weber, 2001, 2008). However, there is no study to explain the difficulties experienced by the students concerning the proof techniques. Moreover, it was observed that there were limited studies that focused on the proof techniques. In these studies, preservice teachers' proving skills regarding only one special proof technique were investigated. Güler, Özdemir and Dikici (2012) investigated the preservice elementary mathematics teachers' proving skills using mathematical induction. Their study showed that the preservice teachers' proving ability using mathematical induction was low. It was also found that there was a positive and significant relationship between preservice mathematics teachers' views and proving skills using an induction technique. Miral (2013) reported that preservice secondary school mathematics teachers had positive opinions about the necessity of a mathematical proof technique. Imamoğlu (2010) indicated that freshmen in the departments of mathematics, elementary and secondary mathematics education tended to use inductive reasoning when constructing a mathematical proof and most of the senior students tried to use deductive methods as they needed more generalization. Baker (1996) stated that most of the high school and university students participating in his study focused more on the practical aspect rather than conceptual aspect of mathematical induction. Doruk and Kaplan (2018a) determined that preservice elementary mathematics teachers had difficulty in proof by contradiction and proof by contraposition techniques.

When earlier studies were examined in terms of the proof techniques, it was determined that these studies were mostly carried out using the counterexample (Doruk and Kaplan, 2018b; Riley, 2003; Zaslavsky and Peled, 1996) and induction techniques (Baker, 1996; Güler, Özdemir and Dikici, 2012). It was revealed that no study focused on preservice mathematics teachers' ability to determine all the basic proof techniques used for the truth and falsity of propositions. In this sense, it can be stated that there is a need for holistic studies on students' ability to determine the proof techniques. Due to these studies, valuable information can be obtained about students' knowledge, perceptions and difficulties in proof techniques. It is important to determine the students' difficulties in the proof techniques in order to overcome these difficulties. Since proof techniques are implication of logic, it can be said that students' comprehensions about proof techniques are reflected in their reasoning skills. It is clear that the information obtained from these studies will benefit educators who are responsible for teaching proof techniques. Educators can revisit the teaching strategies of the course, taking into consideration the misconceptions and difficulties that arise. The mathematics teacher who will be responsible for the teaching this subject should be informed about the proof methods that are considered to be the spirit (Schoenfeld, 2009) and essence (Ross, 1998) of mathematics.

The aim of this study is to reveal the preservice elementary mathematics teachers' determination skills of the proof techniques. In addition, the arguments used by preservice teachers in determining the proof techniques were examined in order to reveal the preservice teachers' understanding of each proof technique. The following research questions were investigated in the study:

1. How are the preservice teachers' determination skills of proof techniques on integers?
2. What type of arguments do preservice teachers use to determine proof techniques?

Method

Research Model

A qualitative case study approach was adopted in the study. A case study is an in-depth representation and examination of a limited system (Merriam, 2013).

Research Group

The participants of the study consisted of 172 preservice teachers enrolled in the second and third years of the department of elementary mathematics teaching of a state university in the eastern Anatolia region of Turkey. The criterion sampling method was used to select the participants. The basic understanding of the criterion sampling is to work with situations that satisfy a set of predefined criteria. The criteria mentioned herein can be created by the researcher, or a list of previously prepared criteria can be used (Yıldırım and Şimşek, 2011). The criterion used in the study is that preservice teachers having taken the abstract mathematics course in the department of elementary mathematics teaching, in which they were informed about proof techniques. For the first part of the study, each of the participant was given an assigned code identifier; e.g., S1, S2,..., S172. After examining the participants' performance in the Proof Techniques Determination Form (PTDF), five preservice teachers from different achievement levels in the PTDF were randomly selected, and interviews were undertaken. The preservice teachers were asked to reconsider their decisions regarding the proof techniques, and their views on the results of the study were elicited in the interviews. The interviewed preservice teachers were given the pseudonyms of Ayla, Derya, Gaye, Turan, and Senem. Ayla accurately determined the techniques of all proofs in the PTDF while Derya was able to accurately determine five, Senem presented four, Turan three and Gaye only two correct proof techniques. The maximum variation sampling method was used in the selection of the preservice teachers for interviews in order to collect maximum variety of views from them.

Data Collection Tools

In the PTDF, six proofs were constructed using different proof techniques. The preservice teachers' responses were taken in writing. By analyzing the PTDF results, semi-structured interviews were conducted with five preservice teachers with different levels of success. In the development of the PTDF, an opinion was taken from an expert academician having doctoral dissertation on mathematical proof. In the draft form, there were two events for each proof technique. The expert academician examined the activities in the draft form in terms of the validity of the study. He recommended that although the activities in the form were suitable for the study, a single activity for each proof technique should be used since it would be easier to collect and analyze the data. This recommendation was accepted and applied. In the interviews, the performance of the preservice teachers in the PTDF was evaluated with them, and their opinions were noted concerning the proof techniques. Table 1 provides information on the proofs used in the PTDF.

Data Collection

The data of the study were collected in two stages. In the first stage, the preservice teachers' decisions and warrants on the proof techniques were collected from the PTDF. The necessary explanations and sufficient time were given to them. In the second stage, semi structured interviews were carried out with five participants. The interviews were conducted in an environment where the preservice teachers and the researcher could talk to each other, and the effects of external factors were at a minimum level. The interviews were audio recorded.

Table 1. Properties of the proofs in the PTDF

Aim	Proof techniques	Propositions
To demonstrate the truth of the propositions	Proof by contraposition	If n^2 is odd, then n is odd.
	Direct proof	Sum of two odd integers is even.
	Proof by contradiction	Sum of two odd integers is even.
	Proof by induction	Show that $3^n < n!$ is valid for $\forall n \geq 5$
To demonstrate the falsity of the propositions	Counterexample	If n integer is prime, then n is odd.
	Confliction	If m and n are odd, then mxn is even.

Data Analysis

The preservice teachers were asked to determine techniques of proofs in writing and explain why they selected this technique. Content analysis was used to analyze the responses of the preservice teachers. The following criteria were considered in assessing the accuracy of the responses:

- If the preservice teacher wrote the proof techniques correctly, it was evaluated as a correct answer.
- If the preservice teacher misspelled the name of the proof technique but explained the correct technique, this answer was evaluated as a correct answer.
- If the preservice teacher misspelled the name of the proof technique and did not describe the correct proof technique or did not make an explanation, these responses were evaluated as a wrong answer.

As an example of the data analysis, the decision and its warrant of S116 on the proof by contraposition technique is presented below.

> **S116:** Proof was constructed by means of contrast, because we demonstrate that if we take the opposite contrary of the proposition and prove that it is correct as a result of the transaction, we show that the proposition is correct.

S116 did not remember the name of the proof techniques and used a different name. When the explanations were examined, the preservice teacher's expression of "the opposite contrary of the statement" was the logical feature of the proof by contraposition technique. Even if the specified name was incorrect, the answer was evaluated as correct since the explanation was correct. If the preservice teacher had made an explanation that did not reflect the characteristics of the proof by contraposition technique, the answer would have been considered as wrong. The data obtained from the interviews were presented as descriptive.

Findings

Proof Techniques Used to Demonstrate the Truth of Propositions

In this section, four basic techniques used to demonstrate the truth of the propositions were focused. These techniques are: proof by contraposition, direct proof, proof by contradiction, and proof by induction. The preservice teachers' ability to determine these techniques in the proofs that had been made by someone else was sought. In order to reveal preservice teachers' determination skills of the proof techniques, the proposition of "If n^2 is odd, then n is odd" was proven by contraposition and presented to them for evaluating. The decisions of the preservice teachers and their explanations of the technique of this proof were investigated. It was found that the evaluations of the preservice teachers were collected under eight categories as shown in table 2.

Table 2 shows that 36% of the preservice teachers could accurately determine the technique of the proof. However, the preservice teachers had difficulty in determining the proof by contraposition technique. The preservice teachers who could not determine the proof by contraposition technique, mostly stated that this technique was direct proof. These preservice teachers deciding on direct proof technique on the basis that one statement was passed without disconnection to the other, there was step by step progression, there was no contradiction, and the data of the proposition was used directly.

Table 2. Preservice teachers' evaluations of the proof constructed by the contraposition technique

Proof technique	*f*	%
Proof by contraposition	62	*36*
Direct proof	70	*41*
Proof by contradiction	14	*8*
Counterexample	9	*5*
Indirect proof	9	*5*
Trial and error	3	*2*
Wrong proof	2	*1*
No answer	3	*2*

In the second order, preservice teachers confused the contraposition technique with the contradiction technique. In the arguments of the preservice teachers, they stated that it was proof by contradiction as this proof was started in reverse, found to be contradictory, and the inverse of the theorem was found to be contradictory.

Nine preservice teachers indicated that the proof technique was counterexample. The preservice teachers often did not give explanations. Those that did stated that n is odd in the proposition, but later in the proof it is accepted as even and accepted that a reverse theorem was used against the theorem. Nine preservice teachers stated that the proof technique was an indirect proof, but they did not specify the characteristics of proof by contraposition. These preservice teachers stated that the conclusion of the theorem was accepted as even, they stated that because it was based on a similar theorem, there was an indirect proof.

Three preservice teachers indicated that the preposition was proved by trial and error. One preservice teacher gave no explanation. Other preservice teachers commented that since both odd and even numbers were tested in the proof and an even number was tried, and "it was true for an odd number", they preferred this technique. Five preservice teachers did not specify the proof technique with two of them claiming that the proof was constructed wrongly, and the other three preservice teachers did not explain anything. A sample from each category in Table 2 is as follows:

> **S59:** The proof was constructed by the contraposition technique, because this is a resolving technique utilizing the expression, $p \Longrightarrow q$ is $q' \Longrightarrow p'$.
> **S105:** The proof was made using the direct proof technique, reaching an equality by utilizing previous equality.
> **S141:** The proof was constructed by using proof by contradiction, because n^2 was first accepted as even, but the result was odd. It was found to be wrong.
> **S62:** The proof was constructed by using counterexample technique, because n was accepted as odd, but n was shown to be even.
> **S54:** The proof was constructed through the indirect proof technique, because the proof was made using a similar theorem.
> **S112:** The proof was constructed using the trial and error technique, because both an odd and an even number were tried as values, and the result was achieved.
> **S67:** I think this proof was constructed wrongly, because after admitting that n is even and showing that n^2 is even, there is no way that if n^2 is odd, then n is odd. I think it may not always be the right technique to prove that it is the odd one after stating that it is even.

The preservice teachers who had achieved different levels of success from PTDF were asked to re-evaluate the proof. In the interviews, it was revealed that the preservice teachers had insufficient knowledge of the proof techniques, did not undertake a thorough examination, and confused the proof by contraposition technique with the direct proof and proof by contradiction techniques. The preservice teachers' views are presented below.

> **Ayla:** When I read the proposition, it came directly to my mind that it was the proof by contraposition technique. It was the inverse of the proposition. That is, if n is even, then n^2 is even. So, this is proof by contraposition.
> **Derya:** We proved the reverse of proposition, but we found the right result. That's why I said it was proof by contraposition.

Gaye I intentionally did it wrong. I didn't really look at that question. The exact opposite was written. I don't understand why I wrote direct proof; if n^2 is odd, then n is odd but it went wrong because n was even. Now I would say that's proof by contradiction. Are contradiction and inconsistent two different terms? We found an exact opposite example… That's why it is contradiction.
Turan: …I didn't think about it clearly. What would I say if I looked at it now? [reexamining the proof]. I think I would say it is true. Direct proof.
Senem: I said direct proof, but I don't know if that is true. I just wanted to say that. That is what I thought. There is no logic.

Secondly, the proposition of "sum of two odd integers is even integer" was proved by direct proof technique. The preservice teachers were asked to evaluate the proof. Their assessments were collected under eight categories. The information about the evaluations of the preservice teachers are presented in table 3.

Table 3. Preservice teachers' evaluations of the proof constructed by direct proof technique

Proof technique	*f*	%
Direct proof	136	*79*
Trial and error	8	*5*
Proof by contraposition	6	*3*
Proof by induction	3	*2*
Proof by contradiction	1	*0.5*
Indirect proof	5	*3*
Wrong proof	1	*0.5*
No answer	12	*7*

Table 3 shows that most of the preservice teachers had little trouble in determining the direct proof technique. They preferred this technique because the result was obtained by means of inferences, especially from those given directly in the proof. 14% of the preservice teachers incorrectly cited the proof techniques and 7% of them did not respond. Most of the preservice teachers that failed to determine the correct proof techniques indicated that the proof technique was trial and error or proof by contraposition. These preservice teachers stated that they preferred the trial and error technique because the proof was constructed by determining two odd numbers and trying them in the proposition. Those preservice teachers who thought that the technique was proof by contraposition often did not make an explanation. Those that did give an explanation indicated that proof by contraposition was more appropriate and gave this as the logical explanation.

Three preservice teachers decided on proof by induction making generality with two numbers. Five preservice teachers preferred indirect proof as the odd number was represented with different variables, and a variable was changed. A preservice teacher stated the technique was proof by contradiction as it was initially admitted to be contrary to the proposition. Another preservice teacher stated that since the proof had to be made by not trying values, the proof had been constructed incorrectly. Samples of the preservice teachers' explanations for each category are presented below.

S105: The proof was constructed by means of the direct proof technique, because $p \Longrightarrow r$ was obtained from the propositions of $p \Longrightarrow q$ and $q \Longrightarrow r$.
S9: The proof was constructed by the trial technique, because the sum of two odd numbers was an even number. So, it was proven by the trial proof technique.
S50: It was proved by the contradiction technique, because in this technique, instead of $p \Rightarrow q$, it is necessary to prove $q' \Rightarrow p'$.
S112: The proof was constructed by the induction technique, because a generalization was made based on two numbers.
S171: The proof was constructed by means the indirect proof technique, because x_1 and x_2 were accepted as odd and 2k + 1 was written instead of x_1 and x_2.
S8: The proof was constructed by means of proof by contradiction, because we prove the proposition by first accepting that it is true.
S122: The proof was constructed by …. [no method given]. Generation is the most sensible one as in the previous technique. Thus, we will prove the proposition given in a general way without trying the value.

During the interviews with the preservice teachers, they were asked to re-evaluate the proof made by the direct proof technique. When the decisions made by the preservice teachers were examined, it was found that they used the data directly and utilized consecutive arguments for the direct proof technique. It was determined that the change of the variable evoked the proof by the contraposition technique. It was also understood that a superficial examination was performed. The preservice teachers' views are presented below.

> **Ayla:** It was shown directly that they were both odd numbers and their sum is an even number. So, it was a direct proof.
> **Derya:** At first, it was written they were odd numbers. It was shown where the values came from, how they were written. Then, it was added both numbers and the proposition was proven directly.
> **Gaye:** I think it was a direct proof because the proof was made by using givens in the propositions.
> **Turan:** This is the same as the previous question [He previously mentioned proof by contraposition]. x_1 was coded as 2k+1, and x_2 as 2t+1. When we put these numbers in brackets, as k+t gives an integer, the sum of x_1 and x_2 is an even number.
> **Senem:** I don't have any justification for my answer [direct proof].

The proof of "Sum of the number of two odd integers is even integer" was presented to the preservice teachers using the contradiction technique. The preservice teachers were asked to make a reasoned decision for the proof technique. According to the results, the decisions of the preservice teachers were gathered under nine categories. Information about these categories is presented in table 4.

Table 4. Preservice teachers' evaluations of the proof constructed by contradiction technique

Proof technique	*f*	%
Proof by contradiction	62	*36*
Proof by contraposition	64	*37*
Counterexample	14	*8*
Direct proof	12	*7*
Indirect proof	8	*5*
Trial and error	3	*2*
Proof by induction	1	*0.5*
Wrong proof	1	*0.5*
No answer	7	*4*

Only 36% of the preservice teachers were able to determine the correct proof technique. Accordingly, it can be said that most of the preservice teachers had difficulty in determining the proof by contraposition technique. The preservice teachers who did not determine the correct technique mostly confused proof by contradiction with proof by contraposition.

14 preservice teachers stated that the proof technique was counterexample. Since these preservice teachers were generally focused on the opposite of the theorem, they stated that a counterexample was used as the opposite examples were given. 12 preservice teachers indicated that the direct proof was used as examples were given directly or the given expressions were applied directly. Eight preservice teachers stated that the technique was the indirect because variable substitution was performed, and the odd number was accepted instead of the even number in the proof. As two values were used in the proof, three preservice teachers stated that the trial and error technique was appropriate, while one thought it was induction technique without giving any explanation. A preservice teacher claimed that the proof was wrong. The sample statements of the preservice teachers for each category are presented below.

> **S68:** The proof was constructed by means of contradiction, because $\Longrightarrow q \equiv (p' \vee q) \Rightarrow (p \wedge q')'$. $p \wedge q'$ was found to be false, and then $(p \wedge q')'$ was true. As $(p \wedge q')'$ is equivalent to $p \Longrightarrow q$, so the truth of the proposition was shown.
> **S74:** The proof was constructed using the contraposition technique, because the inverse of the initial hypothesis was accepted as true and proven: $p \Longrightarrow q \equiv q' \Rightarrow p'$.
> **S33:** The proof was constructed using a counterexample. The falsity of the proposition was proven using a counterexample, which was an example of an inverse proposition.

S72: The proof was constructed by means of the direct proof technique. I think this technique is more appropriate because we can give an example directly without the need for an expression. It would even be easier if all the proofs were explained directly by the proof techniques. There would be no need for another proof technique.
S136: The proof was constructed by means of the indirect proof technique, because it was used x_1=2k+1 and x_2=2t+1. x was written in terms of t.
S112: The proof was constructed by the trial technique, because two values were tried. As a result, there was only one correct result.
S122: The proof was constructed by the …technique [no method given], because it is logically similar with the second question. My ideas have not changed [S122 stated in the second question that making generalization was more appropriate to prove the proposition].

In the interviews, it was revealed that expressions, such as "*whether or not"*, "the contrary", "inverse", and "opposite" led the preservice teachers to think that the proof was constructed by contradiction technique. In the proof, reaching results by considering the data was directly associated with direct proof technique. The preservice teachers also used a superficial examination when evaluating the proof. The views of the preservice teachers are presented below.

Ayla: I thought the same way. I thought it was the best technique. So, I wrote "proof by contradiction". I wrote it because I saw the word "whether".
Derya: I looked at the word first, which was "the contrary". It was accepted inverse of the proposition and found it wrong, and it used the word "the contrary," so I thought it was the proof by contradiction technique.
Gaye: It is a direct proof again because of the direct value given and substitutions. It was reached the result using the given value in the proof. So, it was found an odd value directly in the proof.
Turan: I wrote proof by contradiction, because nothing else came to my mind.
Senem: I think there was a contradiction here. It was written that the sum of two odd numbers is not an even number. So, there might be a contradiction.

It emerged that the preservice teachers gave similar reasons for using proof by contradiction and proof by contraposition, and confused them. In order to clarify this situation, the preservice teachers were asked about the difference between these two techniques, but except for Ayla, they could not provide an explanation. The preservice teachers' views are presented below.

Ayla: Our teachers had said that there was a difference between proof by contradiction and proof by contraposition. We use the equivalence for one of them and use the inverse of the proposition for the other. For example, let a proposition is given. We start with inverse of the proposition. The equivalence of $p \Longrightarrow q \equiv q' \Rightarrow p'$ is given using proof by contraposition. Let's assume that $q' \Rightarrow p'$ is true, then $p \Longrightarrow q$ would also be true. We achieve the proof by contradiction because it is based on what is not true. For example, a p proposition is given. P would also true if it is found that p' is false.
Derya: Proof by contradiction and proof by contraposition are very close to each other, but they also have certain differences. In one, we achieve the proof by focusing on what is not true. In the other, we use contradiction and reverse logic again. We say it is $(p' \vee q)'$ based on the expansion of $p \Longrightarrow q$. In the other method, we reach the proof using the opposite.
Gaye: Proof by contradiction and proof by contraposition are nearly the same. Isn't $q' \Rightarrow p'$ the same thing as $(p' \wedge q)'$?... I think perhaps one of them is the opposite, the other is not. I'm very confused about which one shows what.
Turan: There were differences between proof by contradiction and proof by contraposition. In proof by contraposition, giving an example is not enough to show the falsity of the propositions. There were more steps to be considered and followed. In proof by contradiction, only one example would be enough.
Senem: There are differences between the two, but I don't really know these differences.

Lastly, proposition of "$\forall n \geq 5$ *is for* $3^n < n!$*"* was used. The proof of this proposition was constructed using induction technique and presented to the preservice teachers for their evaluation. It was revealed that the preservice teachers gave three different decisions. Table 5 provides information about the responses of the preservice teachers.

Table 5. Preservice teachers' evaluations of the proof constructed by induction technique

Proof technique	*f*	%
Proof by induction	163	95
Direct proof	2	1
Trial and error	2	1
No answer	5	3

Table 5 shows that almost all the preservice teachers were able to successfully determine the technique as proof by induction. According to this result, it can be said that the preservice teachers did not have any problems concerning this proof technique. Two preservice teachers thought that the proof was constructed using the direct technique, one of whom provided the explanation that each value was tried. Similarly, one of the two preservice teachers that inaccurately identified the technique as trial and error explained that the result could be achieved by trying each value. Extracts of the preservice teacher statements are presented below.

> **S56:** The proof was constructed by the induction technique, because consecutive arithmetic values were used in the proof.
> **S67:** The proof was constructed by the induction technique. Since the value starts from n = 1, p (1), P (k), p (k + 1) and it goes on indefinitely, it is proven to be accurate for all counting numbers. The result is reached based on a generalization.
> **S69:** The proof was constructed by the induction technique, because we're proving its accuracy by moving from parts to the whole.
> **S41:** The proof was constructed by means of the direct proof technique, because all possible values were tried to reach a result.
> **S148:** The proof was constructed by the trial technique, because we have a limited number of possibilities, so we can find the result by trying each.

In the interview, it emerged that the preservice teachers had no difficulty in identifying the proof by induction technique due to its distinctive structure. The preservice teachers' views are presented below.

> **Ayla:** The proof by induction technique has three conditions. It is proof by induction because this example conforms to a direct approach.
> **Derya:** I think this the proof by induction technique. That's how we learned it last year.
> **Gaye:** It is proof by induction, because it goes from specific to general. A value is assumed. At the beginning, 1 can be chosen, but here 5 was assumed. Starting from five, the result is achieved. The technique is clear from its structure.
> **Turan:** It is the proof by induction technique, because it uses k+1 for k.
> **Senem:** It's true for n = 1 for proof by induction. We accept that n = k is correct and we try to confirm it by proving n = k + 1, which is proof by induction, too.

It was noted that the preservice teachers did not have difficulty in applying the technique of induction compared to other techniques. The preservice teachers were asked about the reasons for this situation, and their responses revealed that this was due to its special structure and application of the induction technique in different courses they had attended preservice. Examples of the preservice teacher views are presented below.

> **Turan:** The process is never ending in proof by induction. We assign a value to n; e.g., 1. We find the result. For n = k and k+1, we multiply the right and left sides of the equation to confirm the result. I think it's a more understandable technique because it has logical steps.
> **Senem:** I understand proof by induction very well because I've seen many examples of it in the general math class. I learned it very well in abstract mathematics. That's why I think I understand it well, because it's systematic. Only one example was used in teaching of other proof techniques. Even in the exam of related courses, we only achieved the result based on that example. That could be the reason.
> **Gaye:** … because it's easier. You know what needs to be done from the structure, and the steps are clearer. It's harder to distinguish the other proofs.

Proof Techniques Used to Demonstrate the Falsity of Propositions

Two techniques used to demonstrate the falsity of the propositions were counterexample and confliction. It was aimed to discover how familiar preservice teachers were to these techniques. The first one is "If n integer is prime, then n is odd”. This proposition was proven with the help of a counterexample and presented to preservice teachers for their assessments. It was found that the preservice teachers made evaluations in five different categories (Table 6).

Table 6. Preservice teachers’ evaluations of the proof constructed by counterexample

Proof technique	*f*	%
Counterexample	151	*88*
Direct proof	4	*2*
Proof by contradiction	6	*3*
Trial and error	7	*4*
No answer	4	*2*

As shown in Table 6, most of the preservice teachers had no difficulty identifying the counterexample technique. Preservice teachers agreed that this proof was consteructed using a counterexample. Among the preservice teachers that misidentified the technique, four thought it was direct proof, six stated that contradiction was used, and seven indicated that the technique was trial and error. However, these preservice teachers generally did not provide any explanation. Extracts from the preservice teachers’ explanations are presented below.

> **S60:** The proof was constructed using a counterexample, because if we find an example that doesn't justify a theorem, it means that the proposition is wrong.
> **S68:** The proof was constructed by the counterexample technique, because it gave an example to refute this theorem, and it refuted this theorem.
> **S84:** The proof was constructed using a counterexample, because in this technique, it is enough to give an example that refutes the proof.
> **S112:** The proof was constructed by means of contradiction, because when we assign different values, we obtain different results that contradict with each other.
> **S33:** The proof was constructed by means of the trial and error technique, because by giving an example, it can be proven as true or false.

The interviewed preservice teachers also agreed that the proof was made by the counterexample technique and they did not have any difficulty in this matter as shown in the extracts below.

> **Ayla:** She gave a direct example. Two is even, but not a prime number.
> **Derya:** Here, two is an even number but primes. It was done by counterexample. It was shown that it was mistake.
> **Gaye:** Because when one example is found that doesn't fit, it is a contradiction.
> **Turan:** Here 2 primes but is an even number. It was found wrong because it gave just one example. So, it is the counterexample technique.
> **Senem:** Because a counterexample is given here. That is why I said it is proof by contradiction.

Table 7. Preservice teachers’ evaluations of the proof constructed by confliction technique

Proof technique	*f*	%
Proof by confliction	58	*34*
Direct proof	67	*39*
Proof by contraposition	13	*8*
Counterexample	6	*3*
Indirect proof	6	*3*
Trial and error	4	*2*
Proof by contradiction	3	*2*
Proof by induction	2	*1*
Wrong proof	1	*0.5*
No answer	12	*7*

Finally, the proposition "If m and n are odd, then mxn is even" was proven to be false using the technique of confliction. The proof constructed by confliction was presented to the preservice teachers for their assessments. The preservice teachers made nine different assessments for the proof of this proposition. Table 7 provides information concerning the assessments by the preservice teachers.

Examining table 7, it can be seen that most of the preservice teachers had trouble in determining the technique of the proof presented. 34% of the preservice teachers were able to accurately identify the proof techniques. Most of the preservice teachers stated that the technique was direct proof but they mostly did not offer an explanation for this. Some of the preservice teachers stated that they preferred the direct proof technique because the results were directly obtained, given definitions were used, and inferences were made. Six of the preservice teachers claimed that the proof was constructed using a counterexample; however, none gave an explanation for this. Six preservice teachers stated that the proof was constructed by indirect proof technique, with only two giving an explanation. These preservice teachers stated that indirect proof technique was used because of changing of variables. Four preservice teachers stated that as input variable, trial and error was used, three preservice teachers thought that the proof was contradiction because it was proved according to the logic of contradiction, two preservice teachers considered that induction was used, and only one preservice teacher stated that the proof was constructed wrongly. Extracts from the preservice teachers' expressions for each category are presented below.

> **S79:** The proof was constructed by means of confliction, because the proposition was accepted as it was, and it was proven according to whether or not it contradicted the known values.
> **S125:** The proof was constructed by means of the direct proof technique, because the process was undertaken depending on whether or not m and n are odd. From the values given, the proposition is directly shown to be correct.
> **S108:** The proof was constructed by means of the indirect proof technique, because it uses other terms, such as m, n, 2k + 1.
> **S17:** The proof was constructed by the trial technique. This is because the number has been given a value and the multiplication of these numbers will be equal to the even number.
> **S72:** The proof was constructed by means of contradiction. This technique was used because it can be solved with the equivalence of $p \Longrightarrow q \equiv (p' V q)'$.
> **S12:** The proof was constructed by the induction technique, because it moves from parts to the result.
> **S122:** The proof is constructed by....... technique (no technique specified), because there is no sense in the selection of this proof technique. By giving a counterproof, it could be done in a short way as evidenced by the proof.

In the interviews with the preservice teachers, it was concluded that the preservice teachers' knowledge of the technique of confliction was not adequate with only Ayla being able to explain the confliction technique. Preservice teachers' views are presented below.

> **Ayla:** Now there were two ways of showing that it was wrong. One was counterexample, and the other was ...... [Trying to remember]. I wrote confliction because it didn't come to my mind. There was a confliction in that when an odd number was assumed, the result was an even number. It came out wrong by accepting it right, so I said it was the confliction technique.
> **Derya:** Right now, I think my response is wrong [reexamining the proof; She's trying to make the proof herself]. I found an odd number. This is a false proof, but the way it goes is direct proof. I said indirect, but I think it is wrong. I think it is a direct proof because it was found wrong.
> **Gaye:** This one was proven directly, too because it is shown that m and n are odd numbers; that is to say, I think it was m = 2k + 1.
> **Turan:** I'm changing my response here, direct proof. What is done here is... A property that violates the addition. So, it is a direct proof.
> **Senem:** Because it is like, you know, we found it is nothing. It is a contradiction technique here.

Conclusion, Discussion and Recommendations

As result of the study, it was revealed that the preservice teachers were successful in the determination of direct proof, proof by induction and counterexample techniques. The reason for preservice teachers being successful in recognition of proof by induction and counterexample was that these techniques had distinctive characteristics and forms. The preservice teachers made explanations and gave arguments in this direction in their written responses and interviews. When examining the previous studies on generating counterexamples and constructing proof by induction skills, it was shown that preservice mathematics teachers had difficulty in generating a counterexample and proving by induction (Doruk & Kaplan, 2018b; Güler, Özdemir & Dikici, 2012). In his study, Baker (1996) reported that high school and university students focused more on the operational aspect of the induction technique rather than its conceptual aspect. Accordingly, it is possible to state that the preservice teachers in the current study are successful in recognizing the proof by induction technique and counterexamples, while they have difficulty in differentiating and applying these techniques in constructing proofs.

The preservice teachers frequently used direct proof in the activities of the study. This situation was clearly seen in the participants' responses to the proof by contraposition. It emerged that their reasons for using direct proof were based on over-generalization by analyzing their writing arguments and views in the interviews. For example, there were preservice teachers that considered the criteria that might apply to all proof techniques, such as using the definition of the proof, using the data in the proof, reaching the conclusion without processing errors, and advancing the steps with inferences. Moreover, some of the preservice teachers thought that when the variable was changed within the proof, it would be an indirect proof and that it could not be a direct proof. This was an interesting difficulty for the direct proof technique.

It emerged that the preservice teachers had difficulties in identifying the proof by contradiction and proof by contraposition techniques. They mostly confused contraposition with the direct proof technique but also confused contradiction with contraposition. Doruk and Kaplan (2018a) similarly stated that preservice elementary mathematics teachers could not distinguish between proof by contradiction and proof by contraposition. In that study, in addition to the result of their research, it was revealed that the cause of this difficulty was preservice teachers' inaccurate understanding of proof by contradiction and proof by contraposition. They were not able to distinguish them from each other, and they had over-generalized ideas about direct proof. Using "not to be accepted, not true, are contrary" words for proof by contradiction and proof by contraposition and accepting them as keywords for both techniques caused them to fail to understand the difference between them and caused confusion. This situation was evident in the written responses and views of the preservice teachers. This result obtained from the study was in parallel with research reporting that preservice mathematics teachers had difficulty in selecting the appropriate proof techniques when constructing proofs (Güler, 2013; Moore, 1994; Selden and Selden, 2003). Riley (2003) also reached a conclusion that supports this outcome indirectly. In his study, he showed that 57% of the preservice teachers could construct a valid proof with the direct proof technique. The rate of constructing proofs by the indirect technique was found to be only 39%. Accordingly, it can be said that the preservice teachers' failure to construct indirect proof was reflected in their ability to determine the indirect proof technique. Necessary studies should be undertaken to eliminate these difficulties identified in this study. In the abstract mathematics course where proofs and proof techniques are introduced in the department of elementary mathematics teaching, the proof techniques should be explained in a more detailed way. The main differences between these techniques should be emphasized, especially by considering that preservice teachers can have difficulty in distinguishing between the proof by contradiction and proof by contraposition techniques.

It was found that the preservice teachers failed to determine proof by confliction, which was used to falsify the propositions, and from this, it was revealed that the preservice teachers' knowledge on this subject was inadequate. If it is not known whether a proposition is wrong or is not correct in the technique of finding a conflict, this suggestion is assumed to be correct and some results are obtained without a proposition. If the results obtained contradict with each other or with known, it is concluded that the proposition is wrong (Akkaş et al., 1998). For this reason, studies should be carried out on the understanding of this proof technique by preservice teachers.

When the reasons given by the preservice teachers were examined, it was revealed that they made superficial evaluations. Instead of focusing on the logical structures found in the proof, they focused more on local situations, such as the use of definitions, errors in operations, use of certain words, and changing variables. These results are consistent with studies that found that most university students were undertaking a superficial evaluation while evaluating proofs (Alcock and Weber, 2005; Doruk and Kaplan, 2018a; Morris, 2002; Selden and Selden, 2005). Selden and Selden (2003) stated that preservice teachers focused on superficial errors, such as algebraic expressions and symbolic manipulations instead of general errors; e.g., proving the opposite of the proposition and the basic mathematical gaps in the proof. In this sense, proof evaluation activities should be used in the teaching of related courses with preservice mathematics teachers. Investigating the truth or falsity of a proof constructed by others, discussing the technique of proofs, and finding alternative ways to construct proofs can be used as examples of these activities.

In this study, the preservice elementary mathematics teachers' skills of determining proof techniques on integers were investigated. The data of the study were obtained from preservice teachers writing responses related to six proofs constructed by direct, contradiction, contraposition, induction, counterexample and confliction techniques, and interviews. Similar studies should be undertaken in different areas of mathematics, such as geometry, calculus, and algebra. Since it was revealed that preservice teachers had difficulty in recognizing the proof techniques, further investigation can be designed to determine how preservice teachers use proof techniques and difficulties encountered when using these techniques.

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