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Abstract

Number line models provide a visual aid for students to examine the relationship of integers with each other and facilitate learning of integers and integer operations. Such models are typically used when students are asked real-life problems. This study employs a qualitative case study design to perform an in-depth analysis of how middle grade students use number line models and which models they instinctively prefer for different real-life contexts. To understand students' way of thinking, a questionnaire including real-life context problems about integers was used. Additionally, interview data was used to get a deep insight into students' written responses. The results indicated that middle grade students have three evoked concept images of number line models namely, horizontal, vertical, and the Cartesian plane combining the two. The horizontal number line model was found to dominate over the other two models even in contexts that are typically associated with a vertical one. The study also found that more than half of the students that took part in the study could not properly use any number line model. Finally, the study added to the evidence that there was a symbiotic relationship between arithmetic calculation and using number line in that errors made during one are often corrected with the other.

Introduction

Teaching with contexts and models are essential as students can construct a variety of interpretations about the learning content (Lesh, Post, & Behr, 1987). Real-life contexts such as temperature, elevator, profit and loss, and various number line models are particularly important for students' since they provide rich and integrated knowledge to represent a concept in multiple ways and provide relational understanding between those ways (Van de Walle, Karp, & Bay-Williams, 2010). Number line models are representations that can help students learn positions of numbers relative to each other (Beswick, 2011; Heeffer, 2011). Real-life contexts enable students to understand how mathematics can be related to those contexts (Stephan & Akyuz, 2013). Previous research showed that the tasks that have real-life contexts are important to support students' mathematical learning (Cordova & Lepper, 1996; Walkington, Sherman & Petrosino, 2012). Here it is important to state that real-life contexts do not mean using real-world problems that embed only simple arithmetical operations that do not stimulate thinking and questioning. Real-life contexts should be experientially real or at least imaginable to students, should stimulate thinking, and allow a student to construct a mathematical model (Shanty, 2016). Seegers and Gravemijer (1997) emphasize real-life contexts as learning situations that are experientially real for students. Such contexts should support both mathematical learning and motivation as they foster students' interest, engagement, and mathematical reasoning with possible solution strategies (Walkington, Sherman & Petrosino, 2013).

Integers are taught using real-life contexts in many studies to promote a deep and conceptual understanding (Linchevski & Williams, 1999; Schindler, Hußmann, Nilsson, & Bakker, 2017; Ural, 2016; Whitacre, Bishop, Lamb, Philipp, Schappelle, & Lewis, 2012). Number line models are intertwined with real-life contexts for integers (Akyuz, Stephan, & Dixon, 2012; Gravemeijer, 1994; Stephan & Akyuz, 2013). The research points out that comprehension of integers by students through number line models and making sense of real-life contexts of integers from a mathematical perspective are separate issues (Bobis & Bobis, 2005; Gallardo & Romero, 1999; Murphy, 2011; Van den Heuvel-Panhuizen, 2008; Vlassis, 2004). That often causes students to make sense of integers through giving meaning to signs of operations and numbers and number line models independently. The links between the contexts and the number line models for integers could contribute to teaching integers as such links yield clues about how students think of integers and integer operations. According to Gonsalves and Krawec (2014), “a number line representation of a word problem bridges the gap between linguistic understanding and mathematical understanding by depicting the problem information in an abstract but connected way” (Gonsalves & Krawec, 2014, p.162). This bridge enables us to reveal how students conceptualize the information in the problem via its number line representation.

This study examines middle school students’ usage of number line models while solving real-life context problems involving integers. Ultimately, this study aims to shed further light into the questions that:

- given a real-life problem which number line models students instinctively use,
- whether the use of these models contribute to or hinder their solution of the problems,
- and how educators can capitalize on this information to promote a deeper conceptual understanding of integers.

Number Line Models and Integers

The number line enables students to visualize the position of numbers relative to each other, helps teachers demonstrate calculations, and enables students to develop visual strategies (Bramald, 2000; Heeffer, 2011; Murphy, 2011). One of the pioneering reviews of the number line in literature was made by Ernest (1985) who summarized the ways of using the number line as follows: (1) a number line is a classroom practice which teachers use while teaching the addition and subtraction of single-digit whole numbers, (2) it is seen in written objectives involving addition and subtraction of whole numbers, and (3) it is used while making computations. These three ways of using the number line show that the number line is an essential tool in the teaching and learning of whole numbers.

The number line models are used to help students to order and compute whole numbers, integers, rational numbers, and real numbers (Musser, Burger, & Peterson; 2003; O’Daffer, Charles, Cooney, Dossey, & Schielack, 2008; Shanty, 2016; Van de Walle et al., 2010). Specifically for integers, the number line is commonly leveraged to visualize “going to the negative”. While the notion of negative can be perplexing for new learners, the number line provides a visual aid by indicating a zero-point, visually separating the positive from the negative. As for integers, number line models are also used for rational and real numbers for conceptualizing percentages, ratios, proportions, and differences (Bobis & Bobis, 2005; Deliyianni, Gagatsis, Elia, & Panaoura, 2016; Hamdan & Gunderson, 2017; Patahuddin, Usman, & Ramful, 2018; Treffers, 1991; Tunc-Pekkan, 2015).

The literature shows that both the horizontal and the vertical number line models are used for teaching and learning of integers and integer operations. Students are expected to understand that addition is related to forward movements and subtracting is related to backward movements on the number line (Van de Walle et al., 2010). Resnick and Ford (1981) found that students have “counting all” and “counting on” strategies while making calculations. These strategies were the paths that they produce while transferring the arithmetical representation to the movements on the number line. Similarly, Bramald (2000) discussed the strategies that are required to operate on an empty number line (a number line on which no markings are shown). These strategies include “counting on/counting back by 1s, counting on/counting back in steps, counting on/counting back by 10s, learning and using complements in 10, partitioning single-digit numbers, partitioning two-digit numbers, and compensating (p. 6-7)”. Selter (1998) and Van den Heuvel-Panhuizen (2008) also discussed several benefits of using an empty number line over a number line on which markings are explicit. In particular, Van den Heuvel-Panhuizen (2008) argued that the empty number line facilitates the calculation of whole numbers when using strings of beads by differentiating between measuring numbers and measuring quantity. With the empty number line, students are free to divide the number line into the numbers in a flexible way that comes natural to them.

Gonsalves and Krawec (2014) stressed that a deep understanding of operations is necessary to represent operations on the number line in an accurate way. In accord with this, many researchers address the use of number line for the calculation of whole number operations as well as for the construction of arithmetical strategies with whole numbers (Bramald, 2000; Klein, Beishuizen, & Treffers, 1998; Murphy, 2011; Ryan & Williams, 2007). For example, Ryan and Williams (2007) discussed the number line as a useful tool to support arithmetic solutions of whole number problems. They presented examples of calculation strategies to focus on the ways middle-grade students think when working with whole number operations such as counting all, counting on/counting back in 10s, and partitioning two-digit numbers. Murphy (2011) found that calculation strategies on a number line for whole numbers include the normal jump method, jump further method, and finding the difference method.

Klein et al. (1998) examined two approaches, namely the realistic versus the gradual program design for the procedural and strategic competence of addition and subtraction of whole numbers for second graders. The distinction of the first approach (realistic program design) is that it allows students to produce their strategies by "flexible use of strategies from the beginning of the program by making connections to children's informal strategies" (Klein et al., 1998, p.448). The second approach is more procedurally structured compared to the first approach regarding the size of the numbers used, the content of the questions given to the students, and the way the number line is used. Both of the approaches use the empty number line to support students' procedural competence. The results of the study indicated that a number line is a tool that makes students successful in the operations of addition and subtraction.

These previous studies mostly identify what strategies students have for calculations with whole numbers on a number line and how useful instructions can be designed in which the number line is used as a tool. However, studies on calculation strategies with integers on the number line were limited (with the notable exception of Stephan & Akyuz, 2013 and Beswick (2011)). Existing studies on integers do not compare the advantages of using different number line models for different contexts, but rather focus on developing an instruction using a selected context and number line model (Stephan & Akyuz, 2013). Calculation strategies on a number line with integers are different from those with whole numbers as students need to give meaning to both operations and signs of the numbers while making movements on the number line models. This is the reason that this study focuses on students' use of number line models for integers instead of whole numbers.

Number Line in a National Curriculum

Turkish Ministry of National Education (MoNE) in Turkey make explicit mention of using the number line as part of their teaching and learning objectives (MoNE, 2013). Table 1 shows a summary of number line related teaching and learning objectives up to the 7th grade mathematical curriculum in Turkey, the country in which the current study was conducted. In the Turkish curriculum, the number line is first encountered in an elementary grade (grade 5) and then revisited in middle grades. In elementary grade levels, the curriculum suggests introducing unit fractions, giving a sense of measurement via number line models, and ordering fractions on the models. Additionally, converting integer fractions to compound fractions, and showing fractions with area models of fractions and number line models are taught. Following the elementary grade, students use the number line for making comparison of fractions in the 6th grade of middle-grade levels (MoNE, 2013). In these levels, students are required to explain the meaning of integers on the number line, show absolute values, and order integers on the number line. As seen in Table 1 and Figure 1, Turkish curriculum supports using both horizontal and vertical number lines while introducing addition and subtraction operations with integers. As such, when students complete the 6th grade level, they are expected to know how to interpret integers and display them on the number line and to specify integers and make sense of the absolute value of an integer. Additionally, they are expected to compare and order integers and to perform addition and subtraction operations with integers and solve related problems involving these operations. In line with these, the current study was conducted with 7th grade students who are expected to have accomplished these objectives.

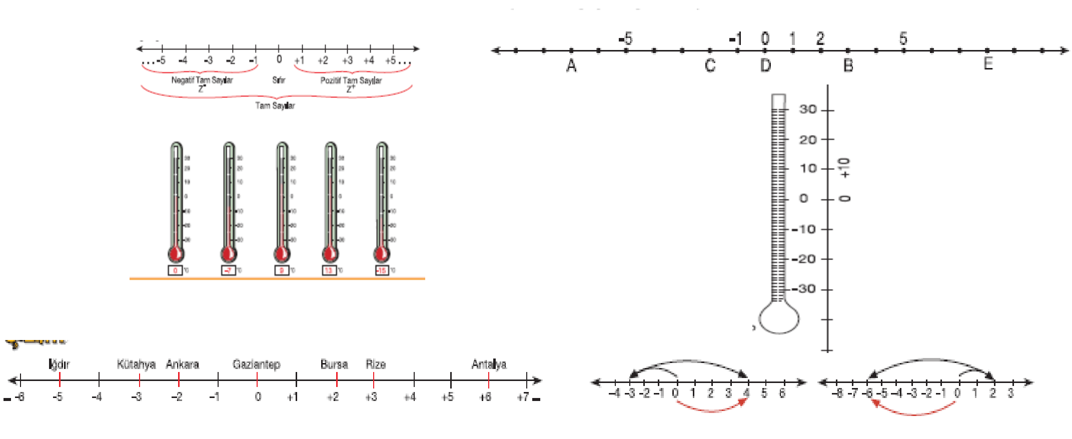


Figure 1. Some examples of number line models in a Turkish textbook (MoNE, 2011)

Table 1. Number line or integers related objectives in middle grade mathematics curriculum (MoNE, 2013)

	Topic	Number line related objectives	Explanation to teachers
5th grade	Decimal representation	Students should be able to show unit fraction on the number line.	
		Students should be able to show numbers given to student as a decimal representation on the number line.	
6th grade	Integers	Students should be able to interpret integers and show them on the number line.	
		Students should be able to specify integers and make sense of the absolute value of an integer.	Focus on what the absolute value means in terms of number line and the real-life context (elevator, thermometer, bank account, etc.).
		Students should be able to compare and order integers.	When comparing, it is emphasized that the large number is to the right of the number line compared to the small number. Integers are included in studies of real-life situations on how to compare and rank them.
		Students should be able to perform addition and subtraction operations with integers; solve related problems about them.	Context such as elevator or thermometer are included in addition and subtraction operations by associating horizontal and vertical number lines.
	Operations with rational numbers	Students should be able to compare and order fractions and to show them on the number line.	The use of appropriate strategies is encouraged when ordering fractions. Strategies that can be used: the closeness of fractions to the whole, being bigger or smaller than half, the closeness of the half, comparison of unit fractions, denominator equalization (consideration of equivalent fractions).

Theoretical Perspective

To examine middle-grade students' concept images of number line models and their calculation strategies on the models Vinner's concept image definition (1991) and Simon's clarification of what a mathematical concept is (2017) are used as theoretical perspectives of the study. In line with these, the role of the perspectives is discussed regarding the purpose of the study.

Tall and Vinner (1981) describe that a concept image includes "all the mental pictures and associated properties and processes" about a concept (p. 152). That introduces several links, and to reveal those links of the networks of concept image of a concept, students' experiences become essential. Vinner (1992) exemplifies those experiences including common experience, typical examples, class prototypes, and so forth existing in a textbook and given in teaching. Regarding this, concept image concerns students' learning of concepts from formally defined ways of them and through experiences in appropriate contexts (Tall & Vinner, 1981). Concept image of students can be encouraged by giving them the opportunity to create their way of thinking and by talking about their way of thinking. It is a network of relationships that is complex and in constant evolution, which is why revealing the concept image of students of a concept is not an easy task for researchers. The highlight is not only on formal concept definition which teachers expect students to learn but also on students' experiences of the same concepts, which the mathematical ideas derived from and those experiences can vary from person to person (Tall & Vinner, 1981; Vinner, 1983; Vinner, 1991). Students' construction of their concept images is typically an ongoing process that includes acquiring or mastering the concept (Vinner & Dreyfus, 1989). Therefore, it is possible to reveal "the portion of the concept image which is activated at a particular time" through (evoked) concept image even though to describe the total cognitive structure of a concept seems to be difficult (Tall & Vinner, 1981, p. 152).

Simon (2017) defines a mathematical concept as "a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (p.123). Regarding the definition of a mathematical concept, he highlights four main points to understand the definition better:

> First, my use of "intended or inferred" indicates that mathematical concept can be used to characterize a learner's understanding (a conception) or to specify a learning goal (an intended conception). Second, researchers may specify different concepts related to the same mathematical relationship. There are diverse ways to characterize understanding in a particular area. Third, the specification of a concept is always in relation to assumptions about prior knowledge of learners involved; coming to know the logical necessity must be based on prior knowledge and available reasoning. Fourth, the specification of the concept is the researchers' characterization of the students' knowledge; it is not meant to capture student language. Anticipating student language may be a useful exercise, but specification of a mathematical concept is done by using the researchers' most accurate language for specifying the learner's understanding of what might be an elementary idea. (p.123).

To put these definitions in the context of the current study, this study examined middle grade students' understanding of the concept of integers regarding the usage of number line models for real-life contextual problems. To understand middle grade students' concept images, not only number line models but also their calculation strategies corresponding to the real-life problems were analyzed. The expected previous knowledge of the students considering the curriculum objectives shown in Table 1 is considered as a sufficient background to make such analysis possible.

Method

Research Design

In this study, a qualitative research design was used with the case study approach which is utilized "to discover meaning, to investigate processes, and to gain insights into an in-depth understanding of an individual, group, or situation" (Lodico, Spaulding, & Voegtle, 2006, p. 269). Working with 7^{th} grade students contribute to an understanding of middle grade students' constructions who have completed the objectives about number line models and integers operations related to these issues. The case study design was employed to gain an in-depth understanding of how the participating students used the number line in their solutions.

Participants

Thirty-two 7^{th} grade students were the participants of the study. We used convenience sampling, defined as "group of individuals who (conveniently) are available for study" (Fraenkel & Wallen, 2005, p. 99), for selecting the participants. In line with this, we conducted this study in a public middle school located in a low-income district in Ankara, Turkey. Based on the school teacher's judgment of students' prior information (their mathematical knowledge, the variety of mathematics grades, and classroom participation), one classroom including thirty-two students was selected in the spring semester of 2013-2014. In addition to this, purposive sampling method, based on the variety of their solution strategies and the voluntary participation, was used to select four students for interviews about their solutions.

Data Collection

In this study, the data were collected through a questionnaire including open-ended questions and semi-structured interview questions. The questionnaire items (i.e., open-ended questions) and interview questions were developed concerning the learning outcomes of Turkish middle-grade mathematics curriculum (MoNE, 2013). Open-ended questions consisted of five questions, which included contexts such as the thermometer, movement, profit-loss, sea level, and historical time (Table 2). Each question required students to represent the given information as an arithmetic sentence. The rationale for using these contexts was that they not only give meaning to integers in real-life contexts but may also evoke imagery of a number line. In this regard, these contexts are considered as appropriate to activate different parts of concept images. Also, middle grade students are familiar with those contexts while learning integers. With the given empty horizontal and vertical number line models, students had an opportunity to answer the open-ended questions in the way that comes natural to them. All of the questions except the historical timeline were prepared based on a student textbook (MoNE, 2011). The historical timeline question was prepared based on utilizing Internet resources.

Table 2. Real-life problems used in the study

Context	Real-life Problem	Explanation
Thermometer	When Dilek looks at a thermometer, she notices that the temperature shows -2°C. The temperature goes up 11°C when the heater starts to burn. In this case, what should be the final temperature of the thermometer? If I show my solution as a number sentence:	The problem concerns how students represent the temperature increase with a number line model. Using the example of the thermometer helps to question the vertical and horizontal number line preferences of students.
Movement	Volkan walks 20 m towards west and then goes 14 m back. Where and in which direction is he, compared to the starting point? If I show my solution as a number sentence:	The last position of the person is required. This question allows us to see how students make movements considering directions.
Profit and Loss	Mustafa has a £60,000 loss per month in the first four months of 2005. Next, for the remaining eight months he has £5,000 profit per month. What is the annual profit and loss situation for him? If I show my solution as a number sentence:	The problem requires students to reason with *large* quantities on a number line. Unlike the other problems, students need to use multiplication to calculate the total profit and total loss. This model is not supported by either the horizontal or vertical number line models.
Sea Level	A submarine is located 150 m below sea level. What is the new location of the submarine when it has lowered 100 m further? If I show my solution as a number sentence:	Students are expected to assign zero to the sea level and to use the negative side of the number line to show the depth.
Historical Timeline	Roman civilization began in 509 BC, and ended in 476 years after Christ. How long did the Roman civilization last? If I show my solution as a number sentence:	The problem represents a scenario that students face in history courses and is typically represented with a horizontal timeline. Students are expected to associate a negative number with the concept of before Christ and mark it on the number line. Students are expected to solve the question by giving the absolute value meaning of the integers as the concept of distance on the number line.

The problems required students to draw a representation of the solution using a number line model (horizontal or vertical) as well as an operational solution to the problem. In other words, students were asked to solve the problem by using either empty horizontal or vertical number lines. All questions required the use of only one operation except for profit and loss context. Accordingly, the solution of each of the problems requires at most two operational transactions that require transforming the meanings in the problems into operations including addition, subtraction, or multiplication.

Before data collection for the main study, we conducted a pilot study with one of the other classrooms of the same teacher. According to students' reactions during problem-solving of the pilot study items, changes to the questions such as the content of the questions, length of questions, alternative understandings about questions, time for completion of the task, and format of the paper were made. Students were given 40 minutes to solve the problems for the main study. After analyzing their solutions to the open-ended questions, interview questions were prepared to elaborate on the written responses of the students. We prepared interview questions by diversification of responses of students who used mostly horizontal, either horizontal or vertical number line models on different questions, or unexpected number line models (such as the Cartesian model). We interviewed them with semi-structured questions which required them to explain their written solutions to open-ended problems., four students were interviewed with questions including such as "What does the second question say? How did you solve?", "Why did you prefer using the horizontal/vertical form of the number line?", and "Do you think the answer to the question is correct? Why?" and so on. The duration of the interview for each student (referred to as B, E, F, and I) was approximately 10 to 15 minutes.

Content validities of the real-life problems and interview questions were assessed based on related objectives in Turkish middle school mathematics curriculum. Furthermore, all the authors of the study two of which are associate level mathematics educators agreed on the appropriateness of the problems and interview questions of the study. Finally, the pilot study gave a further opportunity to revise and improve the questions.

Data Analysis

In this study, we applied content analysis to students' written solutions making logical conclusions by applying certain procedures within a text (Weber, 1990). Students' drawings of horizontal and vertical number lines were examined and grouped by the researchers within the scope of the models in the literature. In line with this, if students had done their solutions by using the horizontal number line, they were categorized as 'horizontal'; if they had used a vertical number line, they were categorized as 'vertical'. If there are other kinds of usage, the most suitable name was given to the representation of the display. Figure 1 shows an example of horizontal and vertical number lines represented in Turkish mathematics textbooks.

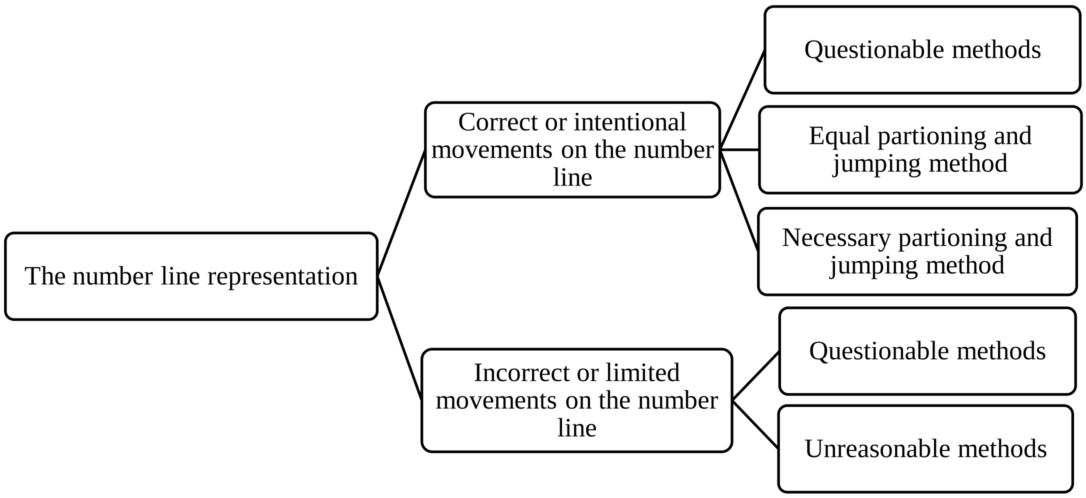
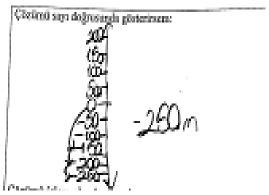
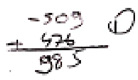
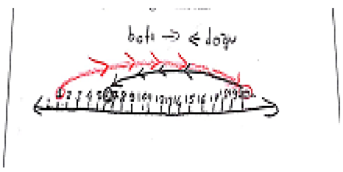
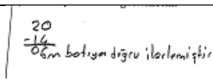
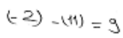


Figure 2. Rubric for evaluating students' strategies

The researchers of the study classified the participants' solution strategies using the methods with whole number calculation on number line models in the literature (e.g., Klein et al., 1998; Van den Heuvel-Panhuizen, 2008). In the beginning, we focused on the movements represented on the number line models. Students' movements showed that number line models could be divided into equal or unequal necessary parts with meaningful and unreasonable movements. In line with this, two calculation strategies were revealed as equal partition and jumping method on the number line and necessary partition and jumping method on the number line. In this regard, it was considered as a clue that the student used the units on the number line as a progress of calculation. Finally, the appropriateness of the calculation strategies on the models was examined to decide whether the participants use the number line for calculation. When a student seemed to represent movements based on the arithmetic result or used the numbers in the question without considering the role of the numbers in the question, the strategies were categorized as questionable or unreasonable movements on the number line, which shows that the number line was not used for calculations. Although unreasonable method represented limited or no representation of movements without meaningful transitions, the questionable method included representations of movements with meaningful problematic transitions. For example, while in the unreasonable method students could draw the number line, they could not represent the solution on the number line. Even they had a representation, it did not have any meaning. However, in the questionable method students tried to represent their solution on the number line with the movements even though they did some mistakes.

To understand the relationship between students' strategies and the arithmetic expressions with the representation of calculation (e.g.: -2+11=9), we examined the correctness of the arithmetical representation of the calculation as well. The correct arithmetical sentence with the correct result was classified as 'correct', while incorrect arithmetic sentence with incorrect product, incorrect arithmetic sentence but the correct result, and correct arithmetic sentence with the incorrect result were classified as an incorrect arithmetical representation for the calculation. Table 3 demonstrates several examples from this process.

Table 3. Sample student responses for data analysis

	Number line representation	Arithmetic result	Rationale for categorization
Equal partition and jumping on the number line	 Correct movements	 Correct arithmetic sentence with correct result	The student made partitioning on the number line with movements of two 50 meter jumps until reaching 250 meters. The way of the drawing is consistent with the movements of the depth as the question requires and the arithmetical sentence of the student was represented correctly.
Necessary partition and jumping on the number line	 Correct movements	 Incorrect arithmetic sentence with correct result	Even though the arithmetical sentence is not totally correct, the student was able to use the number line for the calculation correctly. The student used the number line for calculation representing the absolute value concept on the number line in an accurate way, doing the necessary partition and jumping.
Questionable movements	 Incorrect movements	 Correct arithmetic sentence with correct result	The arithmetical sentence is correct however, the student was not able to use the number line for calculation as he/she started from 1 for the calculation. That makes the method questionable.
Unreasonable methods	 No movements	 Incorrect arithmetic sentence but correct result	The arithmetical sentence of the question is not completely correct and the student just placed the numbers on the number line without making any movements on the number line for calculation. That makes the method unreasonable.

In order to provide inter-rater reliability, firstly, all the researchers analyzed approximately 30% of the data individually, which is adequate to check consistency of coding among the researchers (Neuendorf, 2002). The categories were created based on the related literature (e.g., Klein et al., 1998; Van den Heuvel-Panhuizen, 2008) and the data. After completing the analysis of the data, we had full agreement about categories. Secondly, each researcher analyzed the thirty percent of the data again to check the consistency of coding. More than 80 percent of agreement, which provides an acceptable value of inter-rater agreement, was met among researchers (Miles & Huberman, 1994). Finally, the controversial parts were discussed and clarified until all the researchers reach full agreement of those codings.

Findings

In this section, we first describe the concept images of number line models for different real-life contexts. We then discuss whether students used the number line models as a computational tool or not. Excerpts from the interviews are shared to shed light on students' reasoning for using different number line models.

Concept Images of Number Line Models

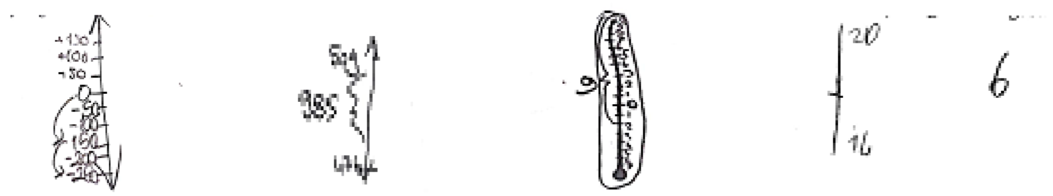
The results showed that students have three mental pictures of number line representations within real-life contexts: horizontal, vertical, and the combination of both number line representations, namely the Cartesian plane. The students predominantly used the horizontal number line model even for contexts that are traditionally taught using the vertical number line. This was followed by the vertical number line model. The Cartesian plane model is used by only a few students and only for the profit and loss context. A summary of these results is presented in Table 4.

Table 4. Students' number line images

Contexts	Typical models in real-life	Students' dominant images	Exceptional models
Thermometer	Vertical	Horizontal	
Movement	Horizontal	Horizontal	
Profit and Loss	No specific model	Horizontal	Cartesian plane model
Sea Level	Vertical	Vertical	
Historical Timeline	Horizontal	Horizontal	

The Vertical Number Line Model

This number line model is a model in which the numbers are located on a number line in a vertical fashion. Some examples from the students' solutions are depicted in Figure 3.



a) Sea level b) Historical timeline c) Thermometer d) Movement

Figure 3. Sample student responses

Some of the students attempted to use a vertical number line model to solve the questions given to them except for the context of profit and loss. As can be seen in Table 5, most of the students used the vertical number line for the sea level context. In all other contexts, the vertical number line was preferred less commonly than the horizontal number line. As for the historical timeline and the movement contexts some students chose to use the vertical number line although these contexts are traditionally associated with the horizontal number line model.

Table 5. The frequency of vertical number line usage

Concept image	Sea level	Historical timeline	Thermometer	Movement	Profit and Loss
Vertical number line	21	7	10	1	0

One of the students was interviewed to explain his reasoning for using the vertical number line in his solutions:

> Researcher: You didn't use horizontal one. Why not? [for the sea level context]
> Student E: Because as it is about depth, vertical comes to my mind. It is a depth, height question.
> Researcher: What do you think when 'vertical' is said to you?
> Student E: Deepness, height, profit-loss, and temperature.

Student E supports the idea that some contexts are matched with specific number line models. In line with this, the sea level question was related to the vertical number line. In contrast to this, the student did not use the vertical number line for the profit-loss context even though he suggested that the vertical number line could be appropriate for this context:

> Researcher: Why did you use horizontal number line here? In which situations do you think the horizontal number line can be used?
> Student E: Horizontal ... length comes to my mind, meter ...
> Researcher: Okay, good, what else?
> Student E: Years are horizontal, for instance 1910, 1920 is like that. Time.
> Researcher: Any other?
> Student E: Profit and loss is vertical but I used horizontal.
> Researcher: Could you use vertical?
> Student E: Yes, I can.

Following the conversation, the student proposed a solution on the vertical number line as shown in Figure 5*:

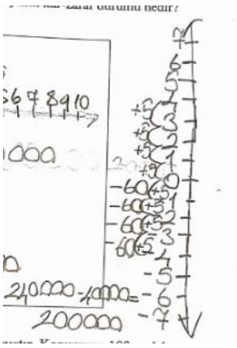


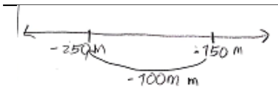
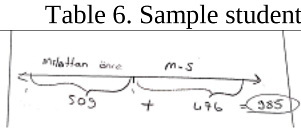
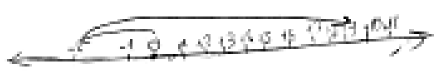
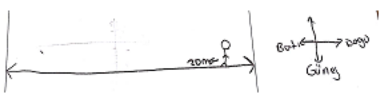
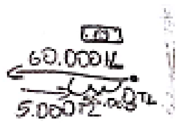
Figure 4. Sample student response
*Note that this solution was proposed by the student during the interviews (not in the initial phase where the students solved the questions individually)

In the solution, the student assigned months for the numbers on the number line while the assets and debts mentioned in the question were the movements. From the sample student responses on the vertical number line (Figures 3 and 4), it is seen that there can be diverse reasons for using different number line models regarding contexts and students could use the vertical number line models in each of the contexts given to them. However, the students did not use the vertical model to represent situations where the vertical model is the typical model in real-life. For example, the thermometer and sea level contexts are conventionally associated with a vertical number line. Participants' initial attempts for solving the problems showed that they had more tendency of using the vertical number line model in the sea level context than the thermometer context.

The Horizontal Number Line Model

In the horizontal number line model, the numbers are ordered from left to right on a horizontal line. In this study, several horizontal number line representations were proposed by students as shown in Table 6.

Table 6. Sample student responses

		
a) Sea level	b) Historical timeline	c) Thermometer
		
d) Movement		e) Profit and Loss

Students used the horizontal number line model to solve each of the problem contexts as seen in the table above. Students placed the negative integers on the left side of the number line while positive integers were placed on the right side. Sometimes, as seen in historical timeline context (Table 6b), students considered the distance without using the negative sign on the left side of the number line model. Additionally, for the profit and loss context, students used the left side of the number line to represent the debt of a person (Table 6e).

Table 7. The frequency of horizontal number line

	Thermometer	Movement	Sea level	Profit and Loss	Historical timeline
Horizontal number line	21	30	11	15	20

The frequency of the horizontal number line usage for each of the real-life contexts are shown in Table 7. As it can be seen in this table, the students were able to use the horizontal number line in all of the contexts. Even for the sea level and thermometer contexts that are typically associated with a vertical number line, many students still preferred to use a horizontal number line. For example, in the interviews student F explained why she preferred to use a horizontal number line in the thermometer context:

> Researcher: Why did you use a horizontal number line? You know we say that you can use one of the number line models and you preferred the horizontal one. For instance, why didn't you prefer using a vertical number line in your solution? Do you remember what you thought at that moment?
> Student F: Well, I thought that the horizontal one was easier to draw.

Student F stated that she preferred the horizontal number line because placing the horizontal model on the paper was easier than doing it with a vertical one. This might be related to the familiarity of drawing with the horizontal number line.

Students' answers related to the contexts of movement and historical timeline, which are supported horizontally in real-life situations, showed that students generally solved the problems by using a horizontal number line. Student F explained her reasoning for the historical timeline context:

> Researcher: Why did you use horizontal number line in your solution?
> Student F: The reason for solving the problem using horizontal number line is that before and after Christ timeline goes like this. I have seen it in social studies course as well.

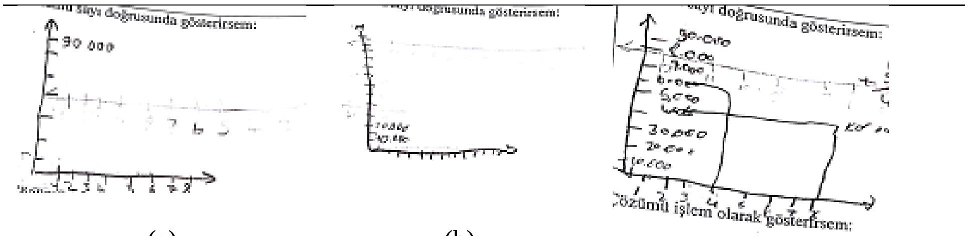
The student's answer showed that the solution with the horizontal number line could be related to the students' familiarity with the model in another course. In this regard, past experiences appeared to serve as a guide for students' concept images. For the movement context, student B explained her reasoning stating that the content of the question is suitable for using the horizontal number line model:

> Researcher: Well, could you solve the problem with the vertical one?
> Student B: I guess I haven't solved the problem using the vertical one because east-west is left and right.

This interview shows that the image of east-west being horizontal on the map is so strong that it compels the students to use a horizontal number line. Indeed, 30 out of 32 students preferred the horizontal number line over the vertical one for this question.

Cartesian Plane Model

This model was created by students combining the positive side of the vertical and horizontal number line models. The model is called the Cartesian plane as students used both the *x* and *y* directions. It was used only in the profit and loss context and by only three students. Different from most of the other contexts, the profit and loss context is not typically associated with a specific number line model. Figure 5 illustrates several examples of this model created by the students.



(a) (b) (c)

Figure 5. Examples of the Cartesian plane model

As can be seen in these examples, the students connected both of the horizontal and vertical number line models to create a Cartesian plane. In these examples, the students assigned the month to the horizontal number line and the money to the vertical number line. In the following dialogue, Student I explains the thinking process for the third solution:

> Researcher: Why did you combine the horizontal with the vertical one?
> Student I: I thought that one month, two months, three months, I am writing the months here (horizontal). I thought that money could be written here (vertical).

In this excerpt, it can be seen that the student used the two dimensions of the Cartesian plane to represent different types of information. This representation was a discovery made by the students as the questions did not explicitly make a mention of such a two-dimensional model. It was interesting to find that students could conceptually combine the two number line models to come up with a new one that is suitable for the profit and loss context. It was also interesting to find that the profit and loss context which has no specific model in real-life is solved by the participants mostly by the horizontal number line model. This context gives us evidence for the tendency of the students to use a horizontal number line model in a context that is new to them.

Blank Answers

Several questions were left unanswered by the students. A summary of these results is shown in Table 8. According to this table, it can be seen that the problem involving the profit and loss context was found to be the most difficult one followed by the historical timeline context.

Table 8. The frequency of blank answers in number line representation

	Thermometer	Movement	Sea level	Profit and loss	Historical timeline
Blank answers	1	1	-	14	5

Calculation Strategies Using the Number Line Models

When the students' answers with number lines and the calculation strategies they used are evaluated, it can be seen that their strategies are based on using number line models as a calculation tool or not.

Table 9. Solutions on number line models

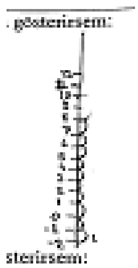
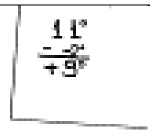
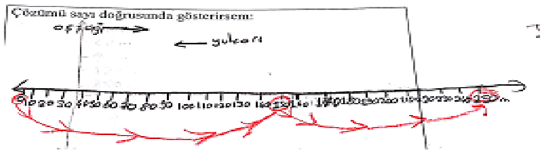
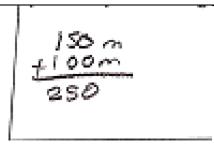
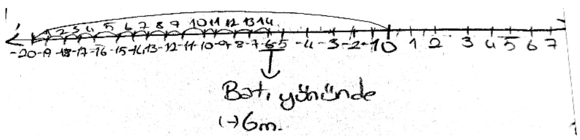
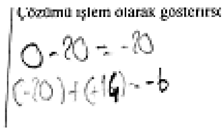
Solution strategies	Thermometer	Movement	Sea level	Profit and Loss	Historical timeline
Calculation tool	17	16	12	2	12
Not a calculation tool	14	15	19	14	15
Blank	1	1	1	14	5
Total	32	32	32	32	32

The number line as a calculation tool includes a solution strategy which represent the calculation on the number line. It also includes incorrect solutions with conceptual movements on the number line (i.e. the final answer is incorrect but the number line is used correctly). On the contrary, *number line as not a calculation tool* category includes inconsistent movements on the number line including two kinds of solution strategies: (1) students' solutions are questionable regardless of the solution being correct or incorrect and (2) students' solutions are unreasonable as number line representations limitedly describe the solution to evaluate whether the number line model is used for calculation. As Table 9 illustrates, most of the students were not able to utilize the number line models to represent calculation.

Number Line as a Calculation Tool

Using number lines as calculation tools requires that students be cognitively involved in the solution process. In this process, number line models are separated into parts and intentional movements are made on the number line to complete the calculation. Although students did not accurately represent the calculation on the number line all the times, the movements represent the steps cognitively integrated leading to the solution of the problem. The students utilized the number line models as a calculation tool by equal partition and jumping method and necessary partition and jumping method.

Table 10. Equal partition and jumping method

Number line representation	Arithmetic representations of students	Expected expression
 (a) thermometer	 Incorrect arithmetic sentence with correct result	-2+(+11)
 (b) sea level	 Correct arithmetic sentence with correct result	-150 + (-100)
 (c) movement	 Correct arithmetic sentence with correct result	-20 + (+14)

In *equal partition and jumping method*, students made purposeful movements on the number line. Students partitioned the number line in equal parts and made jumps on the number line. They created solutions by placing numbers on the number line, leaving small equal parts between the numbers, and then using jumps between the numbers. For example, the range of numbers displayed by students on the number line matches the degrees that appear in the thermometer problem (Table 10a), depth mentioned in the sea level problem (Table 10b), and steps in the movement problem (Table 10c).

As shown in Table 10a, students made single jumps to find an increase in temperature from -2°C to 9°. Although the arithmetic sentence of the student was incorrect, the number line was used in a correct manner, which led to the correct final answer. Alternatively, in Table 10b, the progress is shown in the form of only two jumps (150 meters and then 100 meters). The numbers are labeled individually suggesting counting by ten. The student did not use the negative sign for representing the location of the submarine, and the representation is consistent with the arithmetical sentence and the result. As seen in Table 10c, the student first made a large jump (-20 meters) and then continued with jumps by ones (14 meters). The representation on the number line correctly matched to the arithmetic sentence of the student.

In these models, students partitioned the number line sometimes by ones and sometimes by tens. Depending on the numbers in the contexts, they made movements with fixed number of increments. They found the results on the number line at the end of the movement. When we examine the students' calculations in their arithmetical sentences, we see the result shown in Table 11.

Table 11. Equal partition and jumping method

Arithmetic solution	Thermometer	Movement	Sea level	Profit and Loss	Historical Timeline
Correct	9	13	3	-	-
Incorrect	8	3	-	-	-
Incorrect sentence with correct result	7	2	-	-	-
Incorrect sentence with incorrect result	1	-	-	-	-
Correct sentence with incorrect result	-	1	-	-	-
Blank	-	-	-	-	-
Total	17	16	3	-	-

As can be seen in Table 11, students who used the equal partition and jumping method solved arithmetical sentences both correctly and incorrectly. The common thing in these correct and incorrect answers is that both answers have the correct result in their arithmetical sentences. In line with this, we hypothesize that students who can find correct answers of the arithmetical sentences can utilize the number line models for calculation with equal partition and jumping method successfully. As seen in Table 10, students sometimes used an incorrect arithmetical sentence but still obtained a correct result.

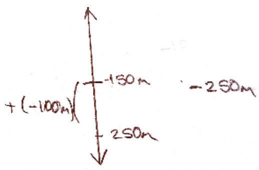
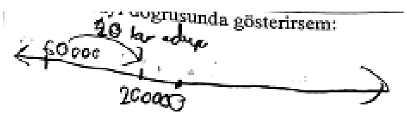
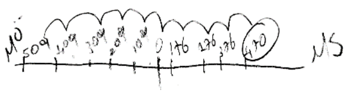
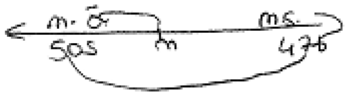
Another method that represents the use of number line model for calculation is *the necessary partition and jumping method*. In this method, students partitioned the number line in unequal parts. In these situations, they used large jumps among numbers that are needed for the calculation. Table 12 represents how students used the necessary partition and jumping method and the corresponding arithmetic sentence they created.

As can be seen in Table 11, students used the method for the sea level, profit and loss, and historical timeline contexts where the numbers are too big to count one by one. Therefore, this solution strategy may be due to the nature of the problem. Table 12a is a solution for the sea level context and can be used to illustrate this method. We see that students located zero invisibly and wrote 150 meters as -150 on the number line. Even though the solution shows that the student did not use the negative sign for the result of the problem on the number line (250 is written instead of -250), it shows that the student correctly represented the move towards the bottom of the sea. The student used the necessary partition locating the numbers with unequal intervals on the number line.

Table 12b is a solution for the profit and loss problem where the students were required to find the annual profit and loss situation for £60,000 loss per month in each of the four months of a year and £5,000 profit for each of the remaining eight months. The student assumed zero as a starting point which he did not spell out visibly (although the tick in the middle of the number line appears to represent zero). He created £60,000 loss representing the number on the negative side of the number line and representing profit by an arrow to the right. The final result was represented to the left of zero indicating a net loss, which is consistent with the arithmetical result. The arithmetical sentence and the result are incorrect because of the misinterpretation of the information given in the problem. Even though the arithmetical representation is incorrect, the student used negative side to represent the loss as a correct way of thinking.

Table 12c illustrates the solution strategies of the student in the historical timeline context. The student wrote zero as a starting point and BC (MÖ in Turkish) for the left side and AD (MS in Turkish) for the right side. The necessary numbers were used for the jumps while representing the calculation on the number line. Another figure, Table 12d, illustrates that students correctly established the arithmetical sentence despite not being able to get the correct result. They used the number line for calculation correctly considering the necessary movements from BC to AD. Similar to this example, in Table 12e, the student represented the location of the submarine below zero with no use of negative sign, probably considering the distance of the submarine to the sea level. She used the arithmetical sentence in an accurate way using the negative sign to represent the location of the submarine, but failed to find out the result with respect to the arithmetical sentence. Instead she wrote the result obtained on the number line model.

Table 12. Necessary partition and jumping method

Number line representation	Arithmetic representations of students	Expected expression
 a) sea level	(-150) + (-100) = (-250) m Correct arithmetic sentence with correct result	-150-100
 (b) profit and loss	5.000 . 8 = 40.000 60.000 − 40.000 = 20000 zarar Incorrect arithmetic sentence with incorrect result	8*5000+ 4*-60.000
 (c) historical timeline	−509 + −476 985 Incorrect arithmetic sentence but correct result	|-509|+476
 d) historical timeline	509 + 476 = 984 Correct sentence with incorrect result	|-509|+476
 e) sea level	−150 + −100 = 250 Correct sentence with incorrect result	-150-100

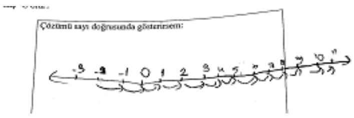
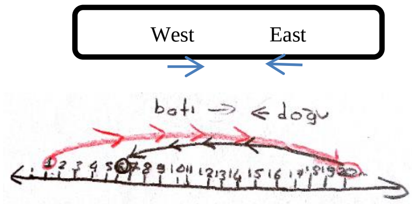
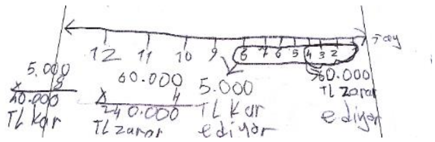
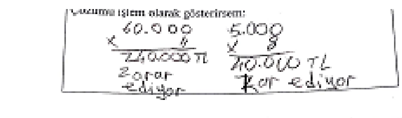
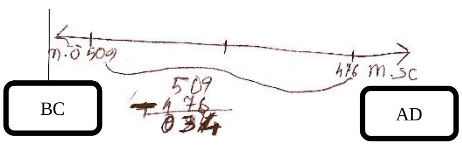
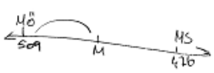
Students who used the necessary partition and jumping method in these contexts exhibited a conceptual understanding of how to use the number line correctly. The sample student responses shown in Table12 show that students can use the number line for calculation by using this method giving meaning to the context as similar to the equal partition and jumping method.

Table 13. Necessary partition and jumping method

Arithmetic solution	Thermometer	Movement	Sea level	Profit and Loss	Historical timeline
Correct	-	-	8	-	5
Incorrect	-	-	1	2	7
Incorrect arithmetic sentence with incorrect product	-	-	-	2	1
Incorrect arithmetic sentence but correct product	-	-	-	-	3
Correct sentence with incorrect product	-	-	1	-	3
Blank	-	-	-	-	-
Total	-	-	9	2	12

When we specifically examine the correctness of the arithmetic solutions and the representation of the solution on the number line, we can say that the result obtained on the number line took precedence over the result obtained using arithmetical sentences. It was often the case that although the arithmetic sentence or the result of this sentence was incorrect, the correct answer was written by the students by inspecting the result on the number line. To this end, using the number line as a calculation tool helps students correct their arithmetical errors.

Table 14. Questionable movements

Number line representation	Arithmetic representations of students	Expression
 a) thermometer	 Correct arithmetic sentence with correct result	-2+11
 (b) movement	 Correct arithmetic sentence with correct result	-20+(+14)
 (c) profit and loss	 Correct arithmetic sentence with correct result	8*5000+4*-60.000
 (d) historical timeline	 Incorrect arithmetic sentence with incorrect result	|-509|+476
 (e) sea level	 Incorrect arithmetic sentence but correct result	-150-100
 f) historical timeline	 Correct arithmetic sentence with incorrect product	|-509|+476

Number Line Not as a Calculation Tool

Some students utilized the number line models not as a calculation tool. They utilized the number line models by equal partition and jumping or necessary partition and jumping method inaccurately because of students' *questionable movements* and *unreasonable methods* on the number line.

Although students were creating number line models, there were doubts about whether they were using it for calculation. In these situations, the solutions on the number line were questionable as students did not correctly represent calculation on the number line and made questionable movements on the number line. The solutions of the problems on the number line include existing deficiencies in associating the given contexts to the accurate number line representation for calculation. Related examples are shown in Table 14.

In the first solution about the thermometer context (Table 14a), the representation of the calculation on the number line was questionable because although the student found the arithmetic result correctly, the student had difficulty in representing this solution on the number line. It is likely that the student counted all the way to 11 degrees instead of counting 11 intervals. A similar situation was experienced by the student in solving the movement problem (Table 14b). The student found the answer to the movement problem on the number line correctly, but she used the number line starting location as one instead of zero. The representation of the calculation on the number line model could be questionable because the number line was used for calculation differently than it should be for the solution of the problem. In both of the situations, students made mistakes while counting the numbers because of having confusions with the use of intervals.

In Table 14c, the student placed months as numbers, profit and loss as an amount of money. The student used the months as numbers on the number line and handled the numbers on the number line as a variable. It is possible to say that the student thought that the months should be positive numbers but placed them as negative integers would be placed from left to right. The word 'loss' in the question may have led him to deal with the negative side of the number line without using the negative sign. In line with this, months were represented on the number line rather than representing the money and the necessary calculation of it. As seen in the figure, the student connected calculations of profit and loss to the arithmetical expressions rather than reasoning with the number line for calculation. That is an example of existing deficiencies in associating the given contexts to the accurate number line representation for calculation.

Even though the representation appearing on the number line was correct, in Table 14d, the movement of attaching the arithmetical calculation to the number line representation (509-476) supported the idea that students made markings on the number line ignoring the idea of absolute value. Besides, some students did not make any markings on the number line, and that is questionable as they were able to find the correct result. For example, one student's drawings for the sea level context (Table 14e) showed that she represented the initial position (150 m below the sea level) and the last position (250 m below the sea level). Such markings are thought to indicate that students used the number line only for marking the end of the process, but not for calculation. Similar to this, in Table 14f, the student represented the movement partially between zero and 509 BC. This representation suggests that the number line is used only for marking the information given in the question instead of using it for calculation.

Table 15. Questionable methods

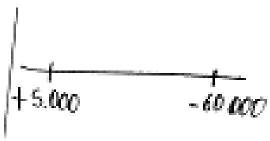
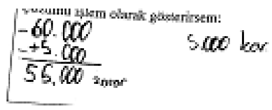
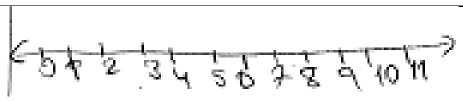
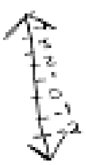
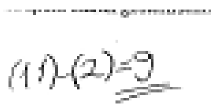
Arithmetic solution	Thermometer	Movement	Sea level	Profit and Loss	Historical timeline
Correct	4	7	2	1	2
Incorrect	3	3	3	6	6
Incorrect arithmetic sentence with incorrect product	1	2	2	6	5
Incorrect arithmetic sentence but correct product	2	-	1	-	-
Correct sentence with incorrect product	-	1	-	-	1
Blank	1	1	3	-	1
Total	8	11	8	7	9

These kinds of drawings have provided the researchers with the evidence that students have difficulty expressing the same information using different representations, interpreting the representations depending on the context, and in transferring a mathematical statement to representation with modeling for the solution. When we specifically examine the students' arithmetical solutions, Table 15 depicts the relation between the correctness of the students' arithmetical representations and questionable movement using on the number line.

Table 15 shows that students who use questionable methods in their number line representations created both correct and incorrect arithmetical representations for the solution of the problems. Incorrect arithmetic sentences of the students vary according to the contexts, which reveals that it might be difficult to interpret the signs of the numbers and the operations according to the meanings in contexts. In other words, students have difficulty in conceptualizing the arithmetic solution on the number line even if the students are procedurally successful in solving the problems.

In the *unreasonable method* category, students' work led us to determine with more certainty that they did not use the number line for calculation; they just drew a line/number line or put some related/unrelated numbers on the number line rather than using it as part of their solution. In this category, the students drew either the vertical or the horizontal number line and merely put numbers on it or made drawings that are very elementary. The following examples in Table 16 are related to the unreasonable movements on the number line category.

Table 16. Unreasonable methods

Number line representation	Arithmetic representations of students	Expression
 (a) historical timeline	 Incorrect arithmetic sentence with incorrect result	|-509|+476
 (b) sea level	 Incorrect arithmetic sentence with incorrect result	-150-100
 (c) profit and loss	 Incorrect arithmetic sentence with incorrect result	8.5000+ 4.-60.000
 (d) movement	 Incorrect arithmetic sentence with incorrect result	-20+(+14)
 (e) thermometer	 Correct sentence with correct result	-2+11

In Table 16a-e, students just drew a horizontal and vertical number line and put numbers on it. No written attempt was observed to solve the problem. Although the students' drawings offer a pictorial representation of the context, they did not continue the solution after placing the numbers mentioned in the questions. The number line was clearly not used as a calculation device in these solutions.

In Table 17, it can be seen that students could perform both correct and incorrect arithmetic solutions when using the unreasonable methods. However, the frequency of the correct responses is reduced compared to that of the students using questionable movements.

Table 17. Unreasonable method

Arithmetic solution	Thermometer	Movement	Sea level	Profit and Loss	Historical timeline
Correct	3	2	2	-	3
Incorrect	3	2	9	6	3
Incorrect arithmetic sentence with incorrect product	2	2	8	6	3
Incorrect arithmetic sentence but correct product	1	-	-	-	-
Correct sentence with incorrect product	-	-	1	-	-
Blank	-	-	-	1	-
Total	6	4	11	7	6

Discussion

In this study, students were asked to solve problems with one positive and one negative integer including calculations with numbers up to five digits. To make number sentences of the addition of two negative integers and to find the distance between two numbers placed on opposite sides of the zero-point, we used real-life context problems. The results of our study reveal students' concept images of number line models within diverse contexts and their solution strategies on number line models.

Our findings indicate that students use three imageries for number line models while solving real-life context problems: horizontal, vertical, and the combination of both number line models termed as Cartesian. Both horizontal and vertical number line models were expected, but surprisingly, one additional model consisting of the combination of them was created by a few participants. Students could conceptually combine the two number line models, use the coordinate axis, and could go beyond the prototype drawing of number line models in the profit and loss context. It was also surprising to find that the profit and loss context which has no specific model in real-life is solved by the participants mostly by the horizontal number line model and no one solved it by the vertical number line model. The vertical number line model was used by the participants in Akyuz et al. (2012)'s research for the profit and loss context. However, in the Turkish curriculum no such usage was observed. Because of this reason, students may prefer to use a horizontal model in a context that is novel to them.

The results of the study have shown that horizontal imagery is the dominant concept image for students. The results of the study have shown that horizontal imagery is the dominant concept image for students. Thermometer, movement, profit and loss, and historical timeline contexts activated the horizontal number line image for students while the sea level context activated the vertical image. Even for the thermometer model, most students preferred to use a horizontal model rather than a vertical one. The students' tendency to use a horizontal model may be due to behavior of a teacher who preferred the horizontal number line because of its boarding layout (Beswick, 2011).

The middle-grade mathematics curriculum in Turkey mentions about the abstract representation of the formal concept of negative numbers and movements on the number line for integers. For example, to visualize the order of integers, the horizontal number line model is used to highlight comparisons. In the curriculum, the number line is represented in middle grades (6-8). In these grade levels, the curriculum does not specify the use of a vertical or horizontal number line for the representation and the suggested textbook (MoNE, 2011) includes both approaches. The vertical number line is used only for thermometer and elevator contexts. The dominance of the horizontal number line in these textbooks might have influenced the students to also predominantly use the horizontal number lines in their solutions.

The reason for why students used the vertical number line in the sea level context whereas they used the horizontal number line in the thermometer context is unclear. That is an unexpected finding and suggests that students still might have dominant concept images depending on the context. This finding might be evidence for the literature suggesting that different contexts can activate different concept images for students (Tall &

Vinner, 1981; Vinner, 1991). Regarding this, real-life situations with different contents might have a role on the number line models used by the students.

A number line is a tool that can be used to represent calculation (Gallardo & Romero, 1999; Van de Walle et al., 2010). Not only addition and subtraction but also multiplication and division on the number line is proposed to be used for learning integers (Van de Walle et al., 2010). In addition to single-digit numbers, it can be used for operations with two-digit numbers as well (Gallardo & Romero, 1999). In this study, students were required not only to solve contextual problems with numbers up to 100 but also numbers above 100. Similar to the results of the several research studies (Klein et al., 1998; Resnick & Ford, 1981; Selter, 1998), in this study students, who can utilize number line as a calculation tool, could make small movements or jumps using different strategies including counting ones, separating the number line in equal or unequal parts supporting to reach the solution for numbers up to 100. Regarding this, the current study confirms previous findings that students tend to make movements regarding the size of the numbers. In addition to this, this study adds additional evidence, which suggests that students have a similar tendency to carry out these kinds of actions on number line models not only to add and subtract whole numbers but also to make operations with integers.

The results of the study have also shown that for thermometer and movement context, almost half of the students and for the other contexts, more than half of the students did not use the number line for calculating. Shanty (2016) emphasizes that students need to understand how to use “the direction of the arrow, the starting point for drawing the arrow, and the difference between two arrows” to apply calculations on number line models accurately (Shanty, 2016, p. 70). Additionally, questionable movements on the number line and unreasonable methods for using the number line clearly show us middle grade students’ misunderstandings about the measurement and linear aspects that underlie the different uses of the number line as a mathematical tool (Gonsalves & Krawec, 2014). Even though the linearity aspect of the number line model seems to be not the main problem for the students, they need to use the number line model giving meaning to measurement on calculating sums and differences of integers. In line with this, the use of the number line as a calculation tool has highlighted the necessity of successful use of the measurement aspect of the number line as well as the use of linearity successfully.

Conclusions and Implications

This study focuses on exploring students' concept images regarding number line models and how they use various number line models in solving integer problems taken from real-life contexts. This study shows us that the students predominantly use the horizontal number line model even in contexts that are epitomized by a vertical number line. It also shows that some students can combine the horizontal and vertical models to create a more sophisticated Cartesian plane model. The findings of the current study suggest that introducing the Cartesian plane model as a combination of the simple number line models may be perceived as more natural by students.

This study argues that whether students use the number line as a calculation tool or not should be considered together with their arithmetic solutions. Arithmetic solutions that are inconsistent with the usage of the number line do not provide evidence that the number line is actually used. As a result of such analysis, it was found that more than half of the students in each context did not use the number line for calculation or left the answers blank. This outcome emerged even though the students were specifically asked to solve the problems using a number line model of their choosing. This suggests that the apparently simple number line representation was not internalized by most of the students.

The tendency of the students to choose the horizontal number line over the vertical one suggests that mathematics teachers spend more time during planning to incorporate sufficient examples in which the vertical number line is used. This may be important as in some contexts using a vertical number line may provide a better visual aid than the horizontal one. Real-life contexts using assets and debts as exemplified by Stephan and Akyuz (2012, 2013) and Akyuz et al. (2012) may provide valuable examples in this direction.

Considering the limited studies about number line models and integers (Shanty, 2016), future work could investigate how number line models play a role on students’ learning of integers and integer operations. In particular, controlled teaching experiments can be conducted in which one group of students are exposed to a horizontal number line model, another to a vertical number line model, and a control group not exposed to any number line model. Their performance in solving integer problems from different real-life contexts can then shed light on the advantages and disadvantages of specific number line models.

One of the interesting results of this study is that the students who make the correct moves while using the number line for calculation obtain more correct results. This suggests that the use of number line as a calculation tool enabled them to reach the correct answer despite being unable to form an arithmetically correct sentence. In other words, procedurally incorrect arithmetical sentences do not always lead to conceptually incorrect expressions on the number line. On the contrary, the students who cannot use the number line for calculation fail to form correct arithmetic sentences and obtain correct results. Further studies may examine whether arithmetic proficiency is a prerequisite for using number line models or whether using the number line model correctly leads to better arithmetic proficiency.

Activities that support students to connect the different representations are essential for the meaningful learning of mathematics as students have a chance to observe the same concept with different representations (Lesh et al., 1987). Thus, the situations that allow the transition among the arithmetic sentence, the number line and the word problem should be designed to encourage students to connect these concepts. In addition, the teacher needs to be careful about the use of the different representations (i.e. arithmetic solutions, a solution with the number line, word problem about a real-life context) since students may not use them meaningfully. In other words, even though students use number lines for modeling their solutions, they may not have the appropriate concept image (Bramald, 2000). Thus, they can have difficulties in solving the problems related to integers. Finally, learning about the different partitioning strategies used by students reveals the necessity of making transition between representation tools.

Limitations

The current study has some limitations. First of all, the results of the current study are limited to the collected data, which include the real-life context of the problems given to students and the questions asked to them during the interviews. Different real-life contexts could have produced different answers. Secondly, this study was conducted in one classroom in Turkey. Different outcomes could have been obtained with students of different cultural and mathematical backgrounds, class participations, and grades. Finally, the data of the study have been discussed and analyzed concerning the theoretical background and the related literature mentioned in the current paper. The interpretation of the results could change by using different theoretical perspectives and methods.

References

Akyuz, D., Stephan, M., & Dixon, J. (2012). The role of the teacher in supporting imagery in understanding integers. *Education and Science, 37*(163), 268-282.

Bobis, J. & Bobis, E. (2005). The empty number line: Making children’s thinking visible. In M. Coupland, J. Anderson, & T. Spencer (Eds.), *Making Mathematics Vital: Proceedings of the 20th Biennial Conference of the Australian Association of Mathematics Teachers* (pp. 66–72). Sydney: AAMT.

Bramald, R. (2000). Introducing the empty number line. *Education*, *28*(3), 5-12.

Beswick, K. (2011). Positive experiences with negative numbers. *Australian Association of Math Teachers, 67*(2), 31–41.

Cordova, D. I. & Lepper, M. R. (1996). Intrinsic motivation and the process of learning: Beneficial effects of contextualization, personalization, and choice. *Journal of Educational Psychology, 88,* 715-730.

Deliyianni, E., Gagatsis, A., Elia, I., & Panaoura, A. (2016). Representational flexibility and problem-solving ability in fraction and decimal number addition: A structural model. *International Journal of Science and Mathematics Education, 14*(2), 397 – 417.

Ernest, P. (1985). The number line as a teaching aid. *Educational Studies in Mathematics, 16,* 411– 424.

Fraenkel, J. R. & Wallen, N. E. (2005). *How to design and evaluate research in education* (6th ed.). Boston: McGraw Hill.

Gallardo, A. & Romero, M. (1999). Identification of difficulties in addition and subtraction of integers in the number line. In F. Hitt, & M. Santos (Eds.), *Proceedings of the 21st International Conference for the Psychology of Mathematics Education, North American Chapter* (pp. 275–282). Mexico: Ohio State University.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education*, *25,* 443–471.

Gonsalves, N. & Krawec, J. (2014). Using number lines to solve math word problems: A strategy for students with learning disabilities. *Learning Disabilities Research & Practice, 29*(4), 160–170.

Hamdan, N. & Gunderson, E. A. (2017). The number line is a critical spatial-numerical representation: Evidence from a fraction intervention. *Developmental Psychology, 53*(3), 587–596.

Heeffer, A. (2011). Historical objections against the number line. *Science & Education, 20*(9), 863–880.

Klein, A. S., Beishuizen, M., & Treffers, A. (1998). The empty number line in Dutch second grades: Realistic versus gradual program design. *Journal for Research in Mathematics Education, 29*(4), 443 – 464.

Lesh, R. A., Post, T. R., & Behr, M. J. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 33–40). Hillsdale, NJ: Erlbaum.

Linchevski, L. & Williams, J. (1999). Using intuition from everyday life in 'filling' the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics, 39*, 131-147.

Lodico, M. G., Spaulding, D. T., & Voegtle, K. H. (2006). *Methods in educational research from theory to practice.* John Wiley & Sons, Inc.: USA.

Patahuddin, S. M., Usman, H. B., & Ramful, A. (2018). Affordances from Number Lines in Fractions Instruction: Students' Interpretation of Teacher's Intentions. *International Journal of Science and Mathematics Education*, *16*(5), 909-928.

Miles, M. B. & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Sage publication.

Ministery of National Education (MoNE) (2013). *Ortaokul matematik dersi öğretim programı [Middle school mathematics curriculum]*. Ankara, Turkey: Milli Eğitim Bakanlığı Yayınları.

Ministery of National Education (MoNE) (2011). *7. Sınıf ders kitabı [7th grade mathematics textbook],* Ankara, Turkey: Milli Eğitim Bakanlığı Yayınları.

Murphy, C. (2011). Comparing the use of the empty number line in England and the Netherlands. *British Educational Research Journal, 37*(1), 147–161.

Musser, G. L., Burger W. F., & Blake E. P. (2003). *Mathematics for elementary teachers: A contemporary approach.* John Wiley & Sons, Inc.

Neuendorf, K. A. (2002). *The Content Analysis Guidebook.* Thousand Oaks, CA: Sage Publications.

O'Daffer P., Charles R., Cooney, T., Dossey J., & Schielack J. (2008). *Mathematics for elementary school teachers* (4th ed.). United States of America: Pearson Education, Inc.

Resnick, L. B. & Ford, W. W. (1981). *The psychology of mathematics for instruction*. Hillsdale, NJ: Erlbaum.

Ryan, J. & William, J. (2007). *Children's mathematics, 4 –15: Learning from errors and misconceptions.* Buckingham: Open University Press.

Schindler, M., Hußmann, S., Nilsson, P., & Bakker, A. (2017). Sixth-grade students' reasoning on the order relation of integers as influenced by prior experience: an inferentialist analysis. https://doi.org/10.1007/s13394-017-0202-x.

Seegers, G. & Gravemeijer, K.P.E. (1997) Implementation and effect of realistic curricula. In M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M. van Lieshout (Eds.), *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures* (pp. 255-272). Utrecht: Utrecht CD Press.

Selter, C. (1998). Building in children's mathematics – A teaching experiment in grade three. *Educational Studies in Mathematics, 36,* 1 – 27.

Shanty, N. O. (2016). Investigating students' development of learning integer concept and integer addition, *Journal on Mathematics Education, 7*(2), 57-72.

Simon, M. A. (2017). Explicating mathematical concept and mathematical conception as theoretical constructs for mathematics education research. *Educational Studies in Mathematics, 94,* 117 – 137.

Stephan, M. & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education, 43*(4), 428-464.

Stephan M. & Akyuz, D. (2013). An instructional design collaborative in one middle school. In Claire Margolinas (Ed.), *Task Design in Mathematics Education. Proceedings of ICMI Study 22* (pp. 511-520). Oxford, United Kingdom: University of Oxford.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies of Mathematics, 12,* 151–169.

Treffers, A. (1991). Didactical background of a mathematics program for primary education, In L. Streefland, (Ed.), *Realistic Mathematics Education in Primary School* (pp. 21 – 56), Utrecht: Freudental Institute.

Tunc-Pekkan, Z. (2015). An analysis of elementary school children's fractional knowledge depicted with circle, rectangle, and number line representations. *Educational Studies in Mathematics, 89,* 419–441.

Ural, A. (2016). 7th grade students' understandings of negative integer. *Journal of Studies in Education*, *6*(2), 170 – 179.

Van de Walle J. A., Karp K. S., & Bay-William, J. M. (2010). *Elementary and middle school mathematics: teaching developmentally* (7th ed.). United States of America: Pearson Education, Inc

Van den Heuvel-Panhuizen, M. (2008). Learning from “didactikids”: An impetus for revisiting the empty number line. *Mathematics Education Research Journal, 20*(3), 6-31.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in ‘negativity’. *Learning and Instruction, 14,* 469-484.

Vinner, S. (1983). Concept definition, concept image, and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14*(3), 293 – 305.

Vinner, S. (1991). The role of definitions in teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65–81). Dordrecht: Kluwer.

Vinner, S. (1992). The function concept as a prototype for problems in mathematics learning. In E. Dubinsky, & G. Harel (Eds.) *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 195–214). USA: MAA.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356 – 366.

Walkington, C., Sherman, M., & Petrosino, A. (2012). “Playing the game” of story problems: Coordinating situation-based reasoning with algebraic representation. *Journal of Mathematical Behavior, 31*(2), 174–195.

Walkington, C., Sherman, M., & Petrosino, A. (2013). Supporting algebraic reasoning through personalized story scenarios: How situational understanding mediates performance. *Mathematical Thinking and Learning, 15*(2), 89-120.

Weber, R. P. (1990). *Basic content analysis* (2nd Edition). Sage Publications, Inc.

Whitacre, I., Bishop, J. P., Lamb, L.L.C., Philipp, R. A., Schappelle, B. P., Lewis, M. L. (2012). Happy and sad thoughts: An exploration of children’s integer reasoning? *Journal of Mathematical Behavior*, *31*, 356–365.

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